Minimal Surfaces: Nonparametric Theory

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The URL for these Beamer Slides: *"Minimal Surfaces: Nonparametric Theory"*

`http://www.math.utah.edu/~treiberg/MinimalNonparametricSlides.pdf`

- J. Oprea, *The Mathematics of Soap Films: Explorations with Maple*©, Student Mathematical Library **10**, American Mathematical Society, Providence, 2000.
- R. Osserman, Minimal Surfaces in $\mathbf{R}^3$, in S. S. Chern, *Global Differential Geometry*, MAA Studies in Mathematics **27**, Mathematical Association of America, Providence, 1989, 73–98.
- R. Osserman, *A Survey of Minimal Surfaces*, Von Norstrand Reinhold, New York, 1969.

4. Outline.

- Nonparametric Surfaces
- Area Minimizing Surfaces Satisfy the Minimal Surface Equation
  - Calculus of Variation and Euler Equation
  - Fundamental Lemma of the Calculus of Variations
- Some Solutions of the Minimal Surface Equation
  - Planes, Scherk's Surface, Catenoid, Helicoid
  - Equation for Minimal Surfaces of Revolution
- Existence and Uniqueness Theorem for Minimal Surface Equation.
- Solutions of Minimal Surface Equation are Area Minimizing
- Comparison of Minimal Surface Equation with Laplace's Equation
  - Maximum Principle
  - Nonsolvability of Boundary Value Problem in Annulus
  - Boundary Value Problem in Punctured Disk has Removable Singularity

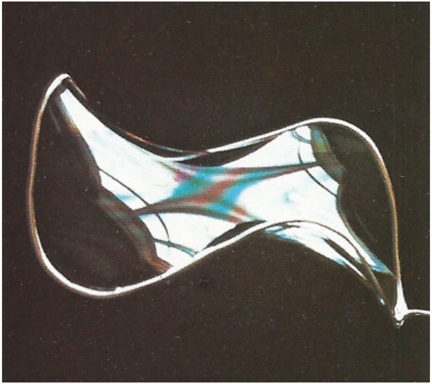
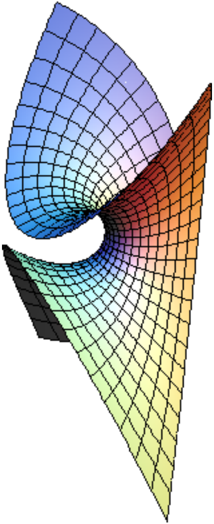


Figure 1: Soap Film Spanning a Wire Loop.

A wire loop dipped in soap solution gives a soap film that spans the wire loop. By surface tension, the film tries to minimize the area being spanned.

A bent square wire frame is spanned by a soap film. One analytic solution is Enneper's Surface, given parametrically for $u, v \in \mathbf{R}$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u^2 - v^2 \\ u - \frac{1}{3}u^3 + uv^2 \\ v - \frac{1}{3}v^3 + vu^2 \end{pmatrix}$$



7. Rotational, Reflection and Dilation Invariance.

We shall be concerned with area minimizing surfaces that have the least boundary area among all surfaces that span the same boundary. Area minimizing surfaces satisfy the minimal surface equation. We will call both minimal surfaces.

Because the area remains unchanged under rotations and reflections, the rotation or reflection of an area minimizing surface is area minimizing.

Because the area is multiplid by a factor c^2 is a surface is dilated by a facrtor c, then the dilation of an area minimizing surface is area minimizing.

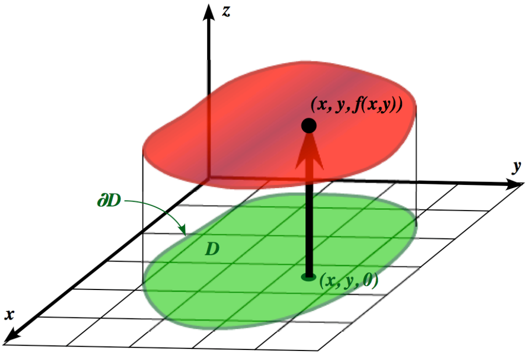


Figure 2: A Nonparametric Surface.

To minimize the geometric detail, we shall discuss surfaces as graphs of functions. Such a surface is called nonparametric.

Such surfaces are determined for domains $D \subset \mathbf{R}^2$ which have smooth boundaries ∂D. They are given as graph of functions.

$$z = f(x, y)$$

for $(x, y) \in D$.

Surface tension in soap films tries to shrink the film spanning a given wire loop. The PDE for soap films characterizes surfaces with minimal area. It is the Euler Equation from the Calculus of Variations.

Let $D \subset \mathbf{R}^2$ be a domain (connected open set) with a piecewise smooth boundary ∂D. The boundary values are given by a function $\psi(x, y) \in \mathcal{C}(\overline{D})$ giving a curve (wire loop) loop in $\mathbf{R}^3$ of boundary values

$$(x, y, \psi(x, y)), \qquad \text{for all } (x, y) \in \partial D.$$

A nonparametric surface spanning these boundary values is given as the graph of a twice continuously differentiable function on D which is continuous on $\overline{D}$,

$$u(x, y) \in \mathcal{C}^2(D) \cap \mathcal{C}(\overline{D})$$

which has the given values on the boundary

$$u(x, y) = \psi(x, y), \qquad \text{for all } (x, y) \in \partial D.$$

10. Area Functional and Minimization Problem.

For simplicity, let us denote the surfaces over D with given boundary values

$$\mathcal{X} = \left\{ u(x,y) \in \mathcal{C}^2(\overline{D}) : \; u(x,y) = \psi(x,y) \text{ for all } (x,y) \in \partial D. \right\}$$

Let $u \in \mathcal{X}$ be a function with given boundary values. The area of the surface is given by the usual area integral

$$A[u] = \int_D \sqrt{1 + |\nabla u(x,y)|^2}\, dx\, dy$$

Note that the smallest value occurs when $\nabla u = 0$ or $u = u_0$ is constant. Then $A[u_0] = |D|$ is the area of D.

Minimization Problem. We seek $w \in \mathcal{X}$ so that the area of the w surface is as small as possible among surfaces with fixed boundary

$$A[w] = \inf_{u \in \mathcal{X}} A[u]$$

Since the $A[u]$ is bounded below by $|D|$, the infimum exists.

Assume that there is $w \in \mathcal{X}$ that has smallest area among all $u \in \mathcal{X}$. If we choose $v \in \mathcal{C}_0^2(\overline{D})$ to be a function with zero boundary values, then for $\eta \in \mathbf{R}$,

$$w_\eta = w + \eta v$$

is another competing surface $w_\eta \in \mathcal{X}$. The function

$$\eta \mapsto A[w_\eta]$$

depends differentiably on η and has a minimum at $\eta = 0$ because $A[w] \le A[w_\eta]$ for all η and is equality at $\eta = 0$. It follows that the first variation of area is

$$\left.\frac{d}{d\eta}\right|_{\eta=0} A[w_\eta] = 0.$$

For a vector function $V(\eta)$ we have $|V(\eta)|^2 = V(\eta) \bullet V(\eta)$ and

$$\frac{d}{d\eta}|V(\eta)|^2 = 2V(\eta) \bullet \frac{d}{d\eta}V(\eta).$$

Hence

$$\begin{aligned}\frac{d}{d\eta}A[w_\eta] &= \frac{d}{d\eta}\int_D \sqrt{1 + \left|\nabla\big(u(x,y) + \eta v(x,y)\big)\right|^2}\, dx\, dy \\ &= \int_D \frac{\nabla\big(u(x,y) + \eta v(x,y)\big) \bullet \nabla v(x,y)}{\sqrt{1 + \left|\nabla\big(u(x,y) + \eta v(x,y)\big)\right|^2}}\, dx\, dy\end{aligned}$$

At $\eta = 0$,

$$\left.\frac{d}{d\eta}\right|_{\eta=0} A[w_\eta] = \int_D \frac{\nabla u(x,y) \bullet \nabla v(x,y)}{\sqrt{1 + |\nabla u(x,y)|^2}}\, dx\, dy$$

For a function v and vector field Z, the Divergence Theorem applied to the vector field vZ yields

$$\int_D \nabla v \bullet Z \, dx \, dy = -\int_D v \operatorname{div}(Z) \, dx \, dy + \int_{\partial D} vZ \bullet \boldsymbol{\nu} \, ds$$

where $\boldsymbol{\nu}$ is the outer normal vector field and ds is the length element of the boundary.

Since $v = 0$ on ∂D, our first variation formula becomes

$$0 = \left.\frac{d}{d\eta}\right|_{\eta=0} A[w_\eta] = -\int_D v \operatorname{div}\left(\frac{\nabla u(x,y)}{\sqrt{1 + |\nabla u(x,y)|^2}}\right) dx \, dy$$

Since $v \in \mathcal{C}_0^2$ can be any function, it follows that the divergence vanishes. (This fact is called the Fundamental Lemma of Calculus of Variations.)

An area minimizer $w \in \mathcal{X}$ satisfies the Minimal Surface Equation

$$\mathcal{M}[u] = \operatorname{div}\left(\frac{\nabla u(x,y)}{\sqrt{1+|\nabla u(x,y)|^2}}\right) = 0. \qquad (1)$$

By carrying out the differentiations there is some cancellation to yield an equivalent form of the Minimal Surface Equation

$$\textstyle\sum_{i,j=1}^{2} a^{ij} u_{ji} \; = \; (1+u_y^2)u_{xx} - 2u_x\, u_y\, u_{xy} + (1+u_x^2)u_{yy} = 0. \qquad (2)$$

This is a nonlinear partial differential equation ($\mathcal{M}[cu]$ may not equal $c\mathcal{M}[u]$) for the area minimizer. It is second order (two or fewer derivatives) and of elliptic type: the coefficient matrix

$$\begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix} = \begin{pmatrix} 1+u_y^2 & -u_x\, u_y \\ -u_x\, u_y & 1+u_x^2 \end{pmatrix}$$

is always positive definite.

Lemma (Fundamental Lemma of the Calculus of Variations)

Suppose $u \in \mathcal{C}(\overline{D})$ is a continuous function that satisfies $\int_D uv \, dx \, dy = 0$ for all $v \in \mathcal{C}_0^2(\overline{D})$. Then $u(x, y) = 0$ for all $(x, y) \in \overline{D}$.

Proof.

By continuitiy at ∂D, it suffices to show $u(x, y) = 0$ for all interior points $(x, y) \in D$. If not, suppose for that this is not true at $(x_0, y_0) \in D$ and that $u(x_0, y_0) > 0$. (Similar argument for $u(x_0, y_0) < 0$.) By continuity, there is a small $r > 0$ so that $u(x, y) > 0$ for all $(x, y) \in B_r(x_0, y_0) \subset D$, a small r-ball about (x_0, y_0) in D. Let $v \in \mathcal{C}_0^2(\overline{D})$ be a "bump function" which is positive $v(x, y) > 0$ if$(x, y) \in B_r(x_0, y_0)$ and $v(x, y) = 0$ otherwise. Then the integral assumption says

$$0 = \int_D uv \, dx \, dy = \int_{B_r(x_0, y_0)} uv \, dx \, dy.$$

However, the integrand in the right integral is positive making the right side positive, which is a contradiction. □

Put $\zeta(\rho) = \begin{cases} (r^2 - \rho)^3, & \text{if } \rho < r^2; \\ 0, & \text{if } \rho \geq r^2. \end{cases}$

$\zeta(\rho)$ is a $\mathcal{C}^2$ function which is positive if $\rho < r^2$ and $\zeta(\rho) = 0$ if $\rho \geq r^2$.

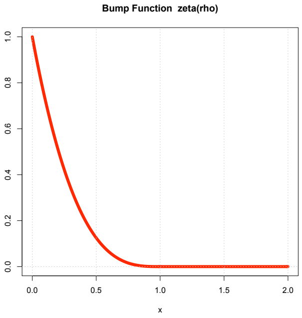


Figure 3: Profile function $\zeta(\rho)$ with $r = 1$.

Figure 4: Bump Function $v(x, y)$ in D.

Sweep the profile around to make a $\mathcal{C}^2$ function that is rotationally symmetric about (x_0, y_0)

$$v(x, y) = \zeta\left((x - x_0)^2 + (y - y_0)^2\right).$$

Planes. If the boundary loop is planar, then the least area spanning surface is planar too. Indeed, if

$$u(x, y) = ax + by + c$$

then all second derivatives vanish $u_{ij} = 0$ so $\mathcal{M}[u] = 0$.

Scherk's Surface. The second easiest to integrate are translation surfaces, namely, those that take the form

$$u(x, y) = g(x) + h(y).$$

The only surfaces that are both minimal and translation are called Scherk's First Surfaces, given by translations and dilations of

$$u(x, y) = \log \cos x - \log \cos y$$

Assuming translation, $u(x,y) = g(x) + h(y)$, the MSE becomes

$$\Big(1 + \dot{h}^2(y)\Big)\ddot{g}(x) + \Big(1 + \dot{g}^2(x)\Big)\ddot{h}(y) = 0.$$

Separating the functions of x from functions of y yields

$$\frac{\ddot{g}(x)}{1 + \dot{g}^2(x)} = -\frac{\ddot{h}(y)}{1 + \dot{h}^2(y)}.$$

The only way that a function of x equals a function of y is if both are constant, say c. We're left with two ODE's. The first is

$$-c = \frac{\ddot{g}(x)}{1 + \dot{g}^2(x)} = \frac{d}{dx}\operatorname{Atn}\big(\dot{g}(x)\big).$$

If $c = 0$ then $g(x) = ax + b$ is linear and $u(x,y)$ is a plane. If $c \neq 0$, this has the first integral for some constant k_1

$$\dot{g}(x) = -\tan(cx - k_1)$$

from which we obtain

$$g(x) = \frac{1}{c}\log\cos(cx - k_1) + a_1$$

The second is

$$c = \frac{\ddot{h}(x)}{1 + \dot{h}^2(x)} = \frac{d}{dx}\operatorname{Atn}\big(\dot{h}(x)\big).$$

This has the first integral for some constant k_2

$$\dot{h}(x) = \tan(cx - k_2)$$

from which we obtain

$$h(x) = -\frac{1}{c}\log\cos(cx - k_2) + a_2$$

Putting them together,

$$u(x,y) = g(x) + h(y) = \frac{1}{c}\log\left(\frac{\cos(cx - k_1)}{\cos(cx - k_2)}\right) + (a_1 + a_2)$$

Note that solutions may be dilated by any factor $c \neq 0$ and translated by any vector $\left(\frac{k_1}{c}, \frac{k_2}{c}, a_1 + a_2\right)$.

20. Scherk's First Surface

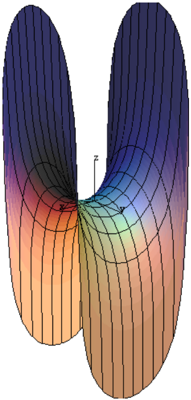
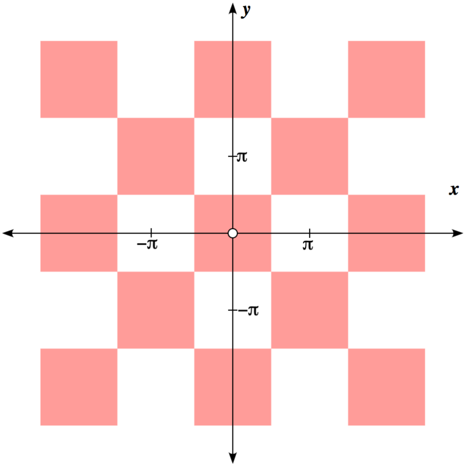


Figure 5: Basic Scherk's First Surface

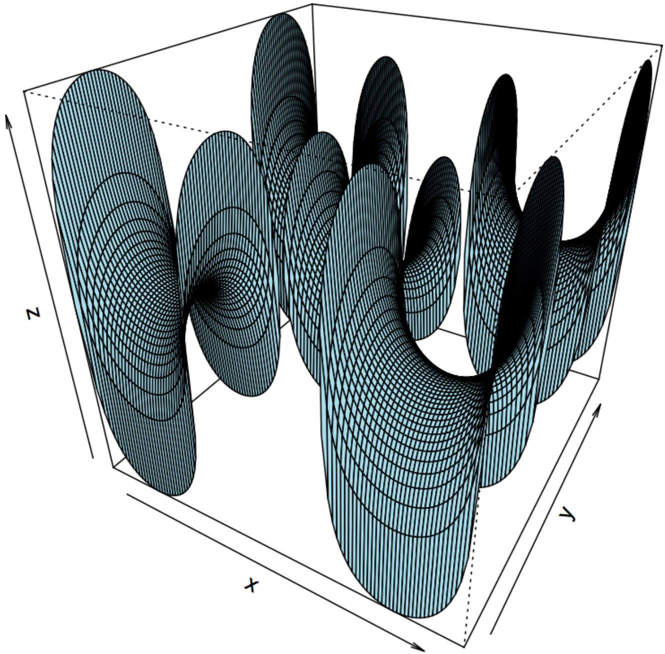
$$z = \log\left(\frac{\cos y}{\cos x}\right)$$

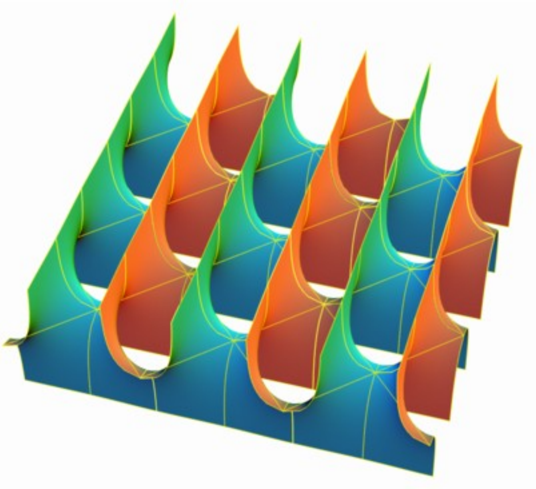
Basic surface on the square $(x, y) \in (-\frac{\pi}{2}, -\frac{\pi}{2}) \times (-\frac{\pi}{2}, -\frac{\pi}{2})$.

Surface continues wherever $\cos y \,/\, \cos x$ is positive, in a checkerboard pattern.



21. Extended Scherk's First Surface





Catenoids. The only minimal surfaces that are rotationally symmetric are the plane and the catenoid. The catenoid is the surface generated by revolving the caternary $r = \cosh z$ about the z-axis. In other words for $r_1 > 0$ constant, let

$$G(r; r_1) = r_a \cosh^{-1}\left(\frac{r}{r_1}\right), \qquad r \geq r_1, \qquad G(r; r_1) \leq 0.$$

The equation

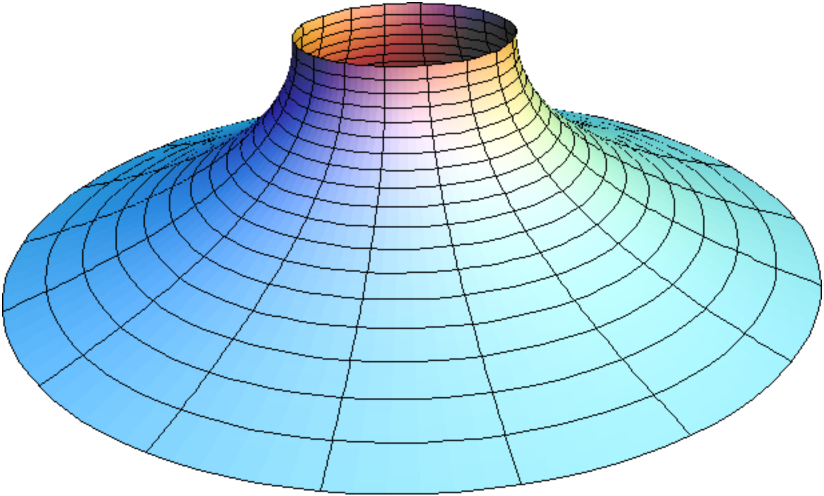
$$z = G(t; r_1), \qquad r = \sqrt{x^2 + y^2}$$

defines the lower half of the catenoid, and is a solution of the minimal surface equation in the entire exterior of the circle

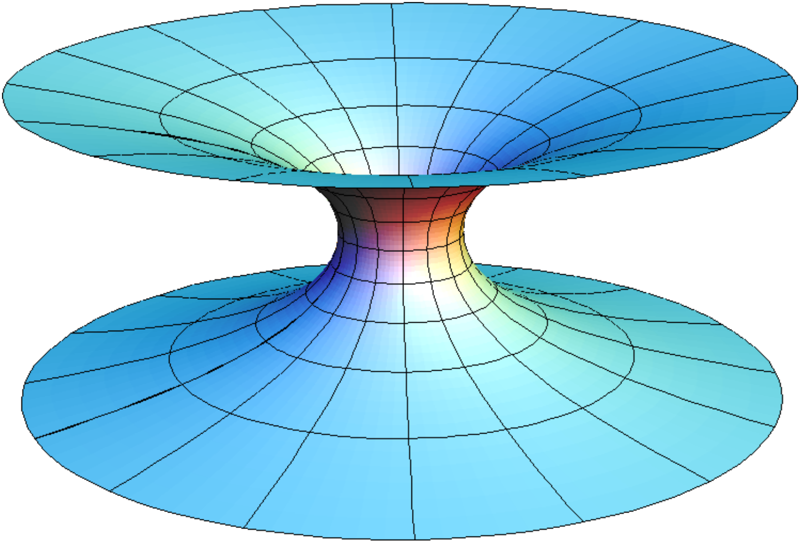
$$x^2 + y^2 = r_1^2.$$

We give two derivations. First we look for solutions of the form $z = G(r)$ and substitute in the MSE. We show that either G is the catenoid or a constant. Second we derive the Euler Equation for minimal surfaces of revolution and show that the catenary is a solution

24. Catenoid Bottom



25. Extended Catenoid



Method 1. Substitute surface of revoloution form into MSE.
We make the ansatz that

$$z = u(x,y) = f(r), \qquad \text{where } r^2 = x^2 + y^2.$$

Then

$$u_x = \frac{\dot{f}x}{r}, \qquad u_y = \frac{\dot{f}y}{r},$$

$$u_{xx} = \frac{\ddot{f}x^2}{r^2} + \frac{\dot{f}y^2}{r^3}, \qquad u_{xy} = \frac{\ddot{f}xy}{r^2} - \frac{\dot{f}xy}{r^3}, \qquad u_{yy} = \frac{\ddot{f}y^2}{r^2} + \frac{\dot{f}x^2}{r^3}$$

so the minimal surface equation becomes

$$\begin{aligned} 0 = {}& \left(1 + \frac{\dot{f}^2 y^2}{r^2}\right)\left(\frac{\ddot{f}x^2}{r^2} + \frac{\dot{f}y^2}{r^3}\right) - \frac{2\dot{f}^2 xy}{r^2}\left(\frac{\ddot{f}xy}{r^2} - \frac{\dot{f}xy}{r^3}\right) \\ & + \left(1 + \frac{\dot{f}^2 x^2}{r^2}\right)\left(\frac{\ddot{f}y^2}{r^2} + \frac{\dot{f}x^2}{r^3}\right) \end{aligned}$$

This simplifies to

$$r\ddot{f} + \dot{f}^3 + \dot{f} = 0.$$

One solution is $\dot{f} = 0$, the plane. Other solutions must have $\dot{f} > 0$ for all r or $\dot{f} < 0$ for all r. If f is a solution then so is $-f$. Separating variables, using partial fractions

$$\frac{\ddot{f}}{\dot{f}} + \frac{\ddot{f}}{1+\dot{f}^2} = \frac{\ddot{f}}{\dot{f}+\dot{f}^3} = -\frac{1}{r}$$

Assuming $\dot{f} > 0$ and and choosing $r_1 > 0$ such that $\dot{f}$ blows up as $r \searrow r_1$,

$$\log \dot{f} - \log \sqrt{1+\dot{f}^2} = \log r_1 - \log r$$

or

$$\frac{\dot{f}}{\sqrt{1+\dot{f}^2}} = \frac{r_1}{r}.$$

Solving for $\dot{f}$ yields

$$\dot{f} = \frac{1}{\sqrt{\frac{r^2}{r_1^2} - 1}}$$

Thus we find that two solutions are

$$f(r) = c_1 \pm r_1 \cosh^{-1}\left(\frac{r}{r_1}\right) \qquad \text{for } r \geq r_1.$$

These give the upper and lower halves of the catenoid. Their union gives a smooth extended minimal surface, called the catenoid.

Method 2. Derive Equation for Minimal Surfaces of Revolution. Let $r = f(z) > 0$ be the radial function for $a \le z \le b$. Then the area of the surface of revolution $x^2 + y^2 = f^2(z)$ is

$$A[f] = 2\pi \int_a^b f(z)\sqrt{1 + \dot f^2(z)}\, dz$$

It suffices to consider only radial variations $f_\eta = f(z) + \eta v(z)$ where $v(a) = v(b) = 0$. Then $\dfrac{d}{d\eta} A[f_\eta] =$

$$2\pi \int_a^b v(z)\sqrt{1 + (\dot f(z) + \eta \dot v(z))^2} + \frac{[f(z) + \eta v(z)]\left(\dot f(z) + \eta \dot v(z)\right)\dot v(z)}{\sqrt{1 + (\dot f(z) + \eta \dot v(z))^2}}\, dz.$$

At $\eta = 0$, integrating the second term by parts, the variation is

$$\left.\frac{d}{d\eta}\right|_{\eta=0} A[f_\eta] = 2\pi \int_a^b \left\{ \sqrt{1 + \dot f(z)^2} - \left(\frac{f(z)\,\dot f(z)}{\sqrt{1 + \dot f(z)^2}} \right)' \right\} v(v)\, dz$$

From the Fundamental Lemma of Calculus of Variations, the Euler equation is

$$\sqrt{1+\dot f(z)^2} - \left(\frac{f(z)\,\dot f(z)}{\sqrt{1+\dot f(z)^2}}\right)' = 0$$

which simplifies to

$$1 + \dot f^2(z) = f(z)\,\ddot f(z).$$

Assuming that $\dot f \neq 0$, observing that the equation doesn't involve z, we apply the usual trick of representing $\dot f$ as a function of f. Assume

$$\dot f = h(f).$$

Then

$$\ddot f = \frac{d}{dz}\dot f = \frac{dh}{df}\frac{df}{dz} = h'(f)\,h(f)$$

which implies

$$1 + h^2 = f\,h\,h'.$$

Separating variables

$$d\log(f) = \frac{df}{f} = \frac{h\,dh}{1+h^2} = d\log\sqrt{1+h^2}.$$

We get a first integral. Assuming that $f = r_1 > 0$ when $h = \dot{f} = 0$,

$$\log(f) - \log(r_1) = \log\sqrt{1+h^2}$$

so

$$\sqrt{\frac{f^2}{r_1^2} - 1} = h = \frac{df}{dz}.$$

Separating variables and substituting $f = r_1 \cosh u$,

$$dz = \frac{df}{\sqrt{\frac{f^2}{r_1^2} - 1}} = \frac{r_1 \sinh u\, du}{\sqrt{\cosh^2 u - 1}} = r_1\, du$$

which means, for some constant z_1,

$$r = f(z) = r_1 \cosh\left(\frac{z-z_1}{r_1}\right).$$

This is a catenary whose throat has radius r_1 and is in the $z = z_1$ plane.

Helicoids. Let us assume that the surface is foliated by horizontal lines through the z-axis

$$z = u(x,y) = f\left(\frac{y}{x}\right)$$

Then

$$u_x = -\frac{\dot f y}{x^2},\ u_y = \frac{\dot f}{x},\ u_{xx} = \frac{\ddot f y^2}{x^4} + \frac{2\dot f y}{x^3},\ u_{xy} = -\frac{\ddot f y}{x^3} - \frac{\dot f}{x^2},\ u_{yy} = \frac{\ddot f}{x^2}$$

so the minimal surface equation becomes

$$\left(1+\frac{\dot f^2}{x^2}\right)\left(\frac{\ddot f y^2}{x^4} + \frac{2\dot f y}{x^3}\right) + \frac{2\dot f^2 y}{x^3}\left(-\frac{\ddot f y}{x^3} - \frac{\dot f}{x^2}\right) + \left(1+\frac{\dot f^2 y^1}{x^4}\right)\frac{\ddot f}{x^2}$$
$$= \frac{1}{x^2}\left[\left(1+\frac{y^2}{x^2}\right)\ddot f + \frac{2y}{s}\dot f\right] = 0$$

In the variable $\zeta = y/x$ we obtain the equation for c_i constants

$$(1+\zeta^2)\ddot f + 2\zeta \dot f = 0$$

Separating variables

$$\frac{\ddot{f}}{\dot{f}} = -\frac{2\zeta}{1+\zeta^2}$$

which has a first integral

$$\log \dot{f} = \log c_1 - \log(1+\zeta^2)$$

or

$$\dot{f} = \frac{c_1}{1+\zeta^2}$$

which leads to the solution called the helicoid.

$$f = c_1 \operatorname{Atn} \zeta + c_2 = c_1\theta + c_2.$$

In other words, z is a linear function of the angle. The surface screws its way up the z-axis. It can be paired with another surface at $\theta + \pi$ which extends the surface smoothly. It is the surface swept out by a line moving up and rotating about the z-axis.

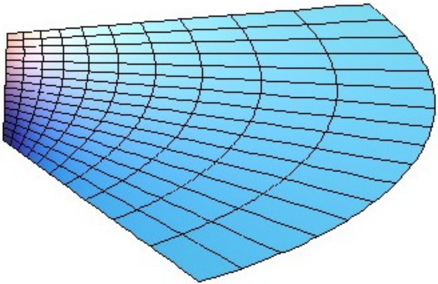
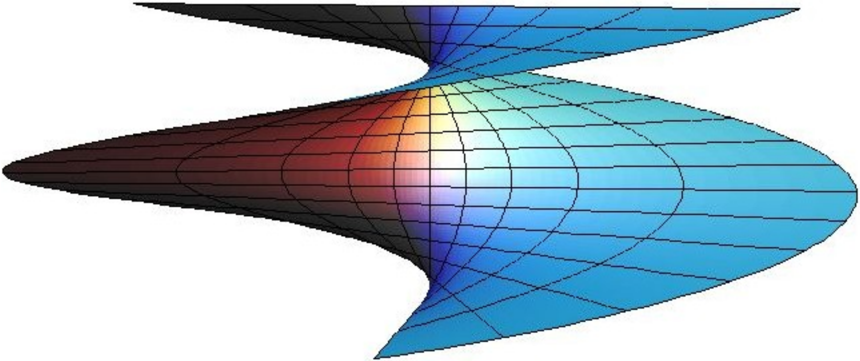


Figure 6: Helicoid: $z = \text{Atn}\left(\frac{y}{x}\right)$ for $0 \le x, y$ and $x^2 + y^2 \le 7$.

35. Extended Helicoid



Theorem (Dirichlet Problem for Minimal Surfaces)

Let $D \subset \mathbf{R}^2$ be a finite domain and S be a surface defined as a graph $z = f(x, y)$ over the domain D.

- *Uniqueness Statement.* *If f satisfies the minimal surface equation*

$$(1 + f_y^2)\, f_{xx} - 2 f_x\, f_y\, f_{xy} + (1 + f_x^2)\, f_{yy} = 0$$

  in D and f extends continuously to the closure of D, then the area of the surface S defined by f is less than the area of any other such surface $\tilde{S}$ defined by a function $\tilde{f}(x, y)$ in D having the same values as f on the boundary of D. Thus there can be at most one solution.

- *Existence Statement.* *[R. Finn] If D is convex, then for every continuous function ψ assigned on the boundary ∂D, there exists a solution of the minimal surface equation in D taking the values ψ on ∂D. Moreover, any domain D with this property must be convex.*

We shall use a calibration argument to show uniqueness.
In the domain $D \times \mathbf{R}$ of $\mathbf{R}^3$, consider the unit vector field

$$V = \left(-\frac{f_x}{W}, -\frac{f_y}{W}, \frac{1}{W} \right) \tag{3}$$

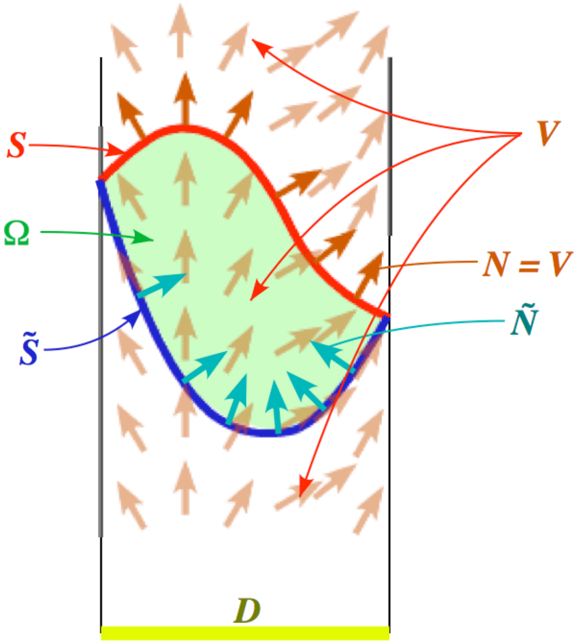
where

$$W = \sqrt{1 + f_x^2 + f_y^2}.$$

Note that V is independent of z so that $\frac{\partial}{\partial z} V = 0$. Hence in $D \times \mathbf{R}$,

$$\operatorname{div} V = -\frac{\partial}{\partial x}\frac{f_x}{W} - \frac{\partial}{\partial y}\frac{f_y}{W} + \frac{\partial}{\partial z}\frac{1}{W} = 0$$

from the minimal surface equation (1).



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Ñ
D

Since S and $\tilde{S}$ have the same boundary, $S - \tilde{S}$ may be viewed as the oriented boundary of the signed open set Ω of $D \times \mathbf{R}$ between S and $\tilde{S}$. Applying the divergence theorem

$$0 = \int_\Omega \operatorname{div} V \, dx \, dy \, dz = \int_{S-\tilde{S}} N \bullet V \, dA$$

where N is the unit normal corresponding to the orientation on $S - \tilde{S}$. By (3),

$$V = N \qquad \text{on } S.$$

Hence

$$A[S] = \int_S N \bullet V \, dA = \int_{\tilde{S}} N \bullet V \, dA \le \int_{\tilde{S}} 1 \, dA = A[\tilde{S}].$$

This inequality is strict unless $V \bullet N = 1$ which would mean f and $\tilde{f}$ have the same gradient at each point so $\tilde{S}$ is a translate of S, and having the same boundary values would have to coincide with S. $\square$

Further appreciation of the minimal surface equation may come by contrasting with solutions of Laplace's Equation

$$u_{xx} + u_{yy} = 0$$

which are called harmonic functions. Laplace's Equation is a linear partial differential equation. It is second order and of elliptic type. It shares many properties of the MSE.

- One common property is that solutions are $\mathcal{C}^\infty$ in the interior of D. Also, if two solutions agree on an open subset, they agree on all of D (unique continuation property).
- A second common property is the uniqueness of solutions of the Dirichlet Problem.
- Consequently, a third common property is the Maximum Principle.

Theorem (Maximum Principle)

Let $D \subset \mathbf{R}^2$ be a finite domain with piecewise $\mathcal{C}^1$ boundary. Let $u, v \in \mathcal{C}^2(D) \cap \mathcal{C}(\overline{D})$ be solutions of the minimal surface equation on D

$$(1 + f_y^2)\, f_{xx} - 2 f_x\, f_y\, f_{xy} + (1 + f_x^2)\, f_{yy} = 0$$

Suppose that there is an inequalty on the boundary $u(x, y) > v(x, y)$ for all $(x, y) \in \partial D$. Then the inequality persists to the whole domain

$$u(x, y) > v(x, y), \qquad \text{for all } (x, y) \in D.$$

So, for example, if $u \le c$ on ∂D then $u \le c$ on D.

Proof. For contradiction suppose there are solutions u and v such that

$$\inf_{\partial D} u - v = \epsilon > 0$$

but $u(x_0, y_0) \le v(x_0, y_0)$ for some interior point $(x_0, y_0) \in D$.

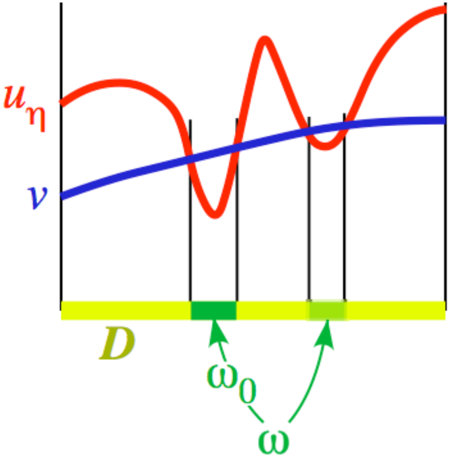


Figure 7: Proof of Maximum Principle

For some $0 < \eta < \epsilon$ the function $u_\eta = u - \eta$ is another solution such that by smoothness,

$$\omega = \{(x, y) \in D : u_\eta(x, y) < v(x, y)\}$$

is an open set with compact closure in D having smooth boundary that contains the point (x_0, y_0). Let $\omega_0 \subset \omega$ be a connected component, so $\omega_0 \subset D$ is a domain with smooth boundary. On this domain, both u_η and v have the same boundary values. By the uniqueness theorem, they must agree on the open set ω_0, hence must agree on D, so $\omega_0 = \emptyset$ which is a contradiction. $\square$

It is not possible to specify general boundary data for the Dirichlet Problem for the MSE on nonconvex domains whereas it is for Laplace's Equation.

An annular domain is for radii $0 < r_1 < r_2$ is

$$\mathcal{A} = \{(x, y) \in \mathbf{R}^2 : r_1^2 < x^2 + y^2 < r_2^2\}$$

The function

$$u(x, y) = a + b \log r, \qquad \text{where } r^2 = x^2 + y^2$$

is harmonic in $\mathcal{A}$ and, for the correct choices of constants a and b, can take any two values on the inner and outer boundaries of $\mathcal{A}$.

The values of solutions of the MSE of on the outer circle of an annulus limit the values on the inner circle.

Theorem

Let $\mathcal{A}$ be an annulus with $0 < r_1 < r_2$. Let $u \in \mathcal{C}^2(\mathcal{A}) \cap \mathcal{C}(\overline{\mathcal{A}})$ be a solution of MSE in $\mathcal{A}$ such that for all (x, y) in the outer circle $x^2 + y^2 = r_2^2$,

$$u(x, y) \le c.$$

Then for all $(x, y) \in \overline{\mathcal{A}}$,

$$u(x, y) \le c + G(r; r_1, r_2), \tag{4}$$

where

$$G(r; r_1, r_2) = r_1 \left(\cosh^{-1} \left(\frac{r_2}{r_1} \right) - \cosh^{-1} \left(\frac{r}{r_1} \right) \right).$$

In particular, the solution is bounded by a quantity depending on c, r_1 and r_2 on the inner circle so cannot be arbitrarily large.

The boundedness on the inner circle is an artifact of the nonparametric formulation and not a geometric restriction. We must devise an argument like the proof of the maximum principle that works without knowing the solution on the inner circle. Using the bottom catenoid as a comparison surface, the basic idea is that its slope on the inner circle is infinite but the slope of the solution is fininte, so that contact cannot occur on the inner circle.

Proof. It suffices to show that a solution $u \le c + G(r; r_1, r_2)$ for all interior points $(x, y) \in \mathcal{A}$, where $r^2 = x^2 + y^2$. Suppose for contradiction that $u > c + G$ at $(x_0, y_0) \in \mathcal{A}$. Let $r_0^2 = x_0^2 + y_0^2$ Consider a slightly smaller annulus with inner circle of radius $r_1 + \epsilon$ where

$$r_0 - r_1 > \epsilon > 0.$$

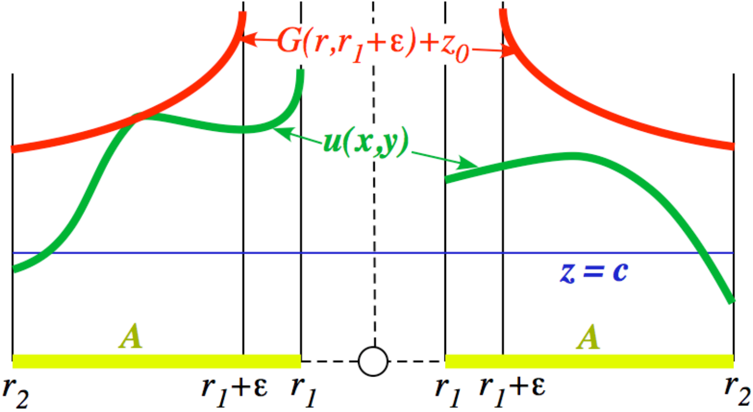
The annulus $\mathcal{A}_\epsilon$ of radii $r_1 + \epsilon$ and r_2 contains the point (x_0, y_0). Since $u \in \mathcal{C}^2(\mathcal{A})$, u and its gradient is bounded on the circle $r = r_1 + \epsilon$.

The lower catenoid on $\mathcal{A}_\epsilon$ is given by

$$G_\epsilon(r) = G(r, r_1 + \epsilon, r_2)$$

Choose z_1 such that $z_1 + G_\epsilon$ passes through $(x_0, y_0, u(x_0, y_0))$, namely $z_1 + G_\epsilon(r_0) = u(x_0, y_0)$. As G_ϵ approximates G, we have $z_1 > c$ for ϵ small enough. We don't know whether $z_1 + G_\epsilon$ exceeds u on $\overline{\mathcal{A}_\epsilon}$. Since u is continuous on $\overline{\mathcal{A}_\epsilon}$, it is bounded. Adding a large enough constant z_0 will put the catenoid above u. Imagine reducing the constant continuously until the surface $z_0 + G_\epsilon$ first touches $z = u(x, y)$ from above. We know that $z_0 \geq z_1$ since we know there is contact at z_1. I claim that the new first touching point (x_1, y_1) is an interior point of $\mathcal{A}_\epsilon$. Because $z_0 \geq z_1$, (x_1, y_1) is not in the exterior circle. Because the gradient of $z_0 + G_\epsilon$ is infinite on the inner circle but that of u is finite, (x_1, y_1) is not in the interior circle either, so must be an interior point.

And now, arguing as in the maximum principle, this is impossible unless $z_0 + G_\epsilon = u$ on all of $\mathcal{A}_\epsilon$, which is a contradiction.

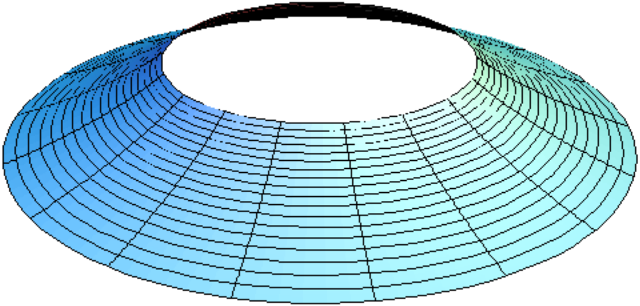


It follows that at (x_0, y_0), and for every small $\epsilon > 0$,

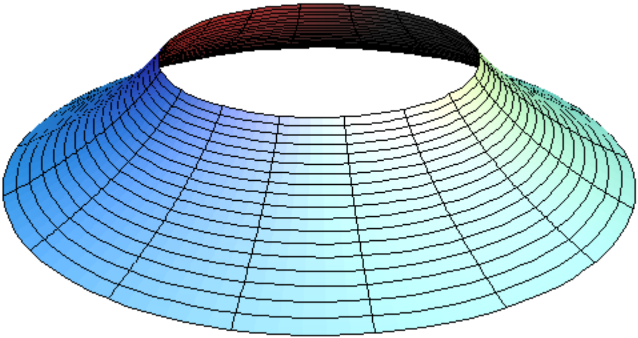
$$u \leq c + G_\epsilon = z_0 + (r_1 + \epsilon)\left(\cosh^{-1}\left(\frac{r_2}{r_1 + \epsilon}\right) - \cosh^{-1}\left(\frac{r_0}{r_1 + \epsilon}\right)\right).$$

Letting ϵ tend to zero proves the estimate (4). $\square$

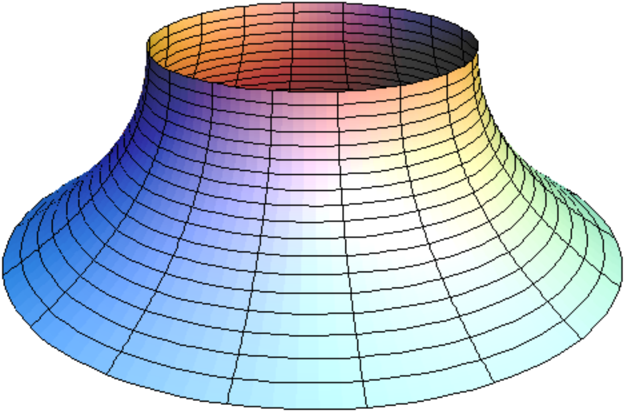
48. Picture of Breakdown of Existence



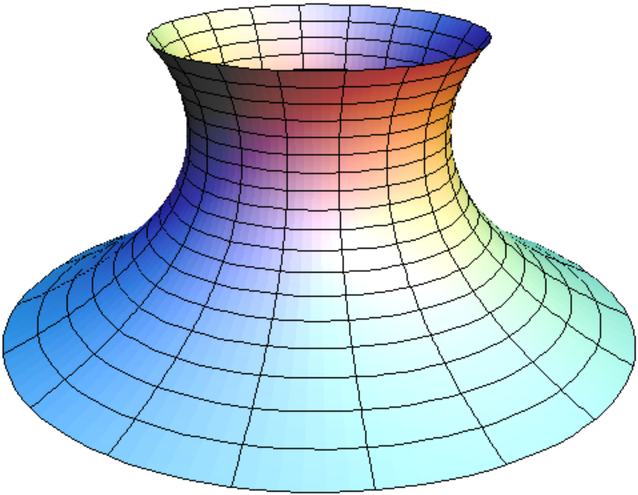
49. Picture of Breakdown of Existence

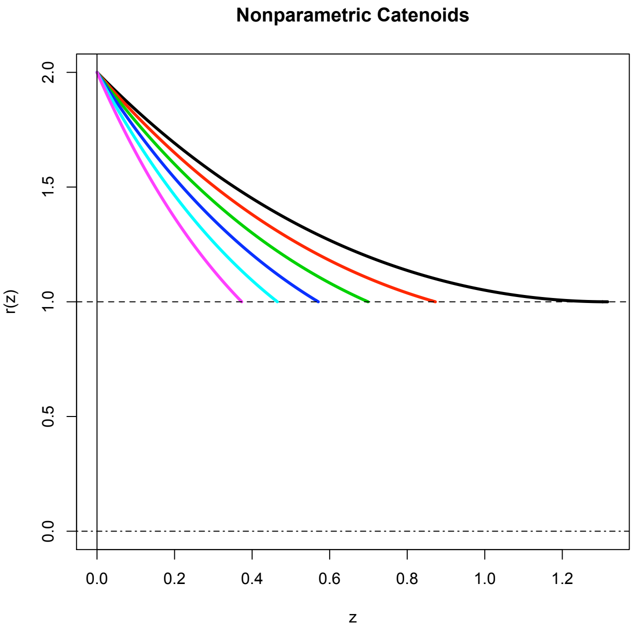


50. Picture of Breakdown of Existence

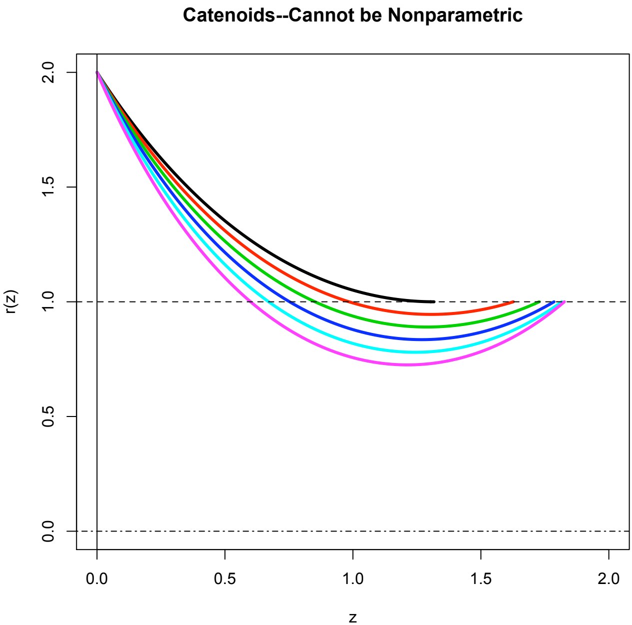


51. Picture of Breakdown of Existence




Nonparametric Catenoids
r(z)
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
z

53. Profile of Catenoids that do not make Nonparametric Surfaces



Here is a second property of minimal surfaces that is not held by solutions of elliptic equations.

Consider the punctured disk

$$\Delta = \{(x, y) \in \mathbf{R}^2 : 0 < x^2 + y^2 \leq 1\}$$

Suppose that $u(x, y)$ is a nonparametric minimal surface defined on Δ which is bounded on the outer boundary $x^2 + y^2 = 1$. Then $u(x, y)$ is bounded on Δ. Moreover, the two-dimensional limit

$$L = \lim_{(x,y)\to(0,0)} u(x, y)$$

exists so that $u(x, y)$ can be extended to the origin by $u(0, 0) = L$ to make a continuous function. In fact, the resulting function is a nonparametric minimal surface on the closed, non-punctured disk. Zero is said to be a removable singularity.

Consider the harmonic function on the punctured disk

$$h(x, y) = \log \frac{1}{r}, \qquad \text{where } x^2 + y^2 = r^2.$$

The function satisfies Laplaces Equation

$$h_{xx} + h_{yy} = 0,$$

is constant $h = 0$ at $r = 1$, but blows up at the origin

$$\infty = \lim_{(x,y)\to(0,0)} h(x, y).$$

Lemma

Let $u(x, y)$ be a nonparametric solution of the MSE on the punctured disk Δ such that $u = 0$ on the outer circle $r = 1$. Then $u(x, y) = 0$ so can be continued to the zero solution of the MSE on the unit disk.

Proof. It suffices to show $u(x, y) \le 0$ for all $(x, y) \in \Delta$. The same argument applied to $-u$ which also solves the MSE shows that $u = 0$. The rest follows from inequality (4). Choose $(x_0, y_0) \in \Delta$ to show $u(x_0, y_0) \le 0$. Let $r_0 = x_0^2 + y_0^2$ and let $0 < \epsilon < 1$. By the upper estimate (4) applied to an annulus whose inner radius is $r_1 = \epsilon$, outer radius $r_2 = 1$ and boundary bound $c = 0$ we get

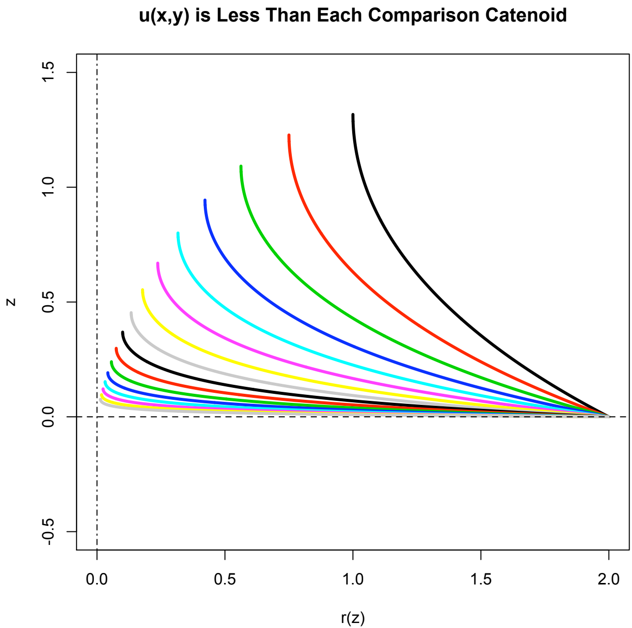
$$u(x_0, y_0) \le G(r_0; \epsilon, 1) = \epsilon \left(\cosh^{-1}\left(\frac{1}{\epsilon}\right) - \cosh^{-1}\left(\frac{r_0}{\epsilon}\right) \right)$$

which tends to zero as $\epsilon \to 0$, proving the lemma. □

Write the inverse hyperbolic cosine as an integral. Then estimate the integrand above by the largest it is in the integral.

$$\begin{aligned}
\epsilon\left(\cosh^{-1}\left(\frac{1}{\epsilon}\right)-\cosh^{-1}\left(\frac{r_0}{\epsilon}\right)\right) &= \epsilon\int_{r_0/\epsilon}^{1/\epsilon}\frac{du}{\sqrt{u^2-1}} \\
&\leq \epsilon\int_{r_0/\epsilon}^{1/\epsilon}\frac{du}{\sqrt{\frac{r_0^2}{\epsilon^2}-1}} \\
&= \frac{1-r_0}{\sqrt{\frac{r_0^2}{\epsilon^2}-1}}
\end{aligned}$$

This tends to zero as $\epsilon \to 0$.


u(x,y) is Less Than Each Comparison Catenoid
z
1.5
1.0
0.5
0.0
-0.5
0.0
0.5
1.0
1.5
2.0
r(z)

Thanks!