Math 2270 § 4. **Fourth Midterm Part 1** Name: Solution

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1. *Find all eigenvalues and eigenvectors. Show your work to get credit.*

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

By expanding the second row the characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & -\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 0 & -\lambda \end{vmatrix}$$
$$= -\lambda(2-\lambda)^2.$$

Thus the eigenvalues are $\lambda_1 = 2$ with algebraic multiplicity two and $\lambda_2 = 0$ with algebraic multiplicity one. We may find the eigenvectors by inspection

$$[\mathbf{0} \quad \mathbf{0}] = (A - \lambda_1 I)[\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{0} = (A - \lambda_2 I)\mathbf{v}_3 = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

All eigenvectors are the nonzero vectors in eigenspaces

$$\mathcal{E}_1 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{E}_2 = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2. *Find a matrix P that makes A similar to a real matrix of the form C. Show that your matrix does the job.*

$$A = \begin{bmatrix} 0 & 5 \\ -1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

First find eigenvalues and eigenvectors. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) + 5$$
$$= \lambda^2 - 2\lambda + 5 = (\lambda - 1)^2 + 4$$

Hence the eigenvalues are $\lambda_1 = 1 - 2i$ and $\lambda_2 = 1 + 2i$. We may find the eigenvector by inspection

$$\mathbf{0} = (A - \lambda_1 I)\mathbf{v}_1 = \begin{bmatrix} -1+2i & 5 \\ -1 & 1+2i \end{bmatrix} \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}.$$

We split the eigenvector into its real and imaginagry parts.

$$\mathbf{u} + i\mathbf{v} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

The similarity making matrix P is made of of real and imaginary parts of the eigenvector. The scaled rotation matrix of the desired form C is made up of real and imaginary parts of the eigenvalue.

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

The easiest way to check is to compute

$$AP = \begin{bmatrix} 0 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = PC.$$

3. *Let $H = \operatorname{span}(S)$ be a subspace of $\mathbf{R}^4$. Show that $\mathbf{b}_1$ is in H. Find additional vectors $\mathbf{b}_2, \mathbf{b}_3, \ldots$ (as many as needed) so that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots\}$ is a basis for H. Explain.*

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}, \qquad \mathbf{b}_1 = \begin{bmatrix} 7 \\ 9 \\ 6 \\ 4 \end{bmatrix}$$

Form the augmented matrix whose columns are S vectors along with $\mathbf{b}_1$ and reduce.

$$[A \,|\, \mathbf{b}_1] = \begin{bmatrix} 1 & 1 & 4 & 1 & 7 \\ 1 & 2 & 5 & 1 & 9 \\ 0 & 2 & 2 & 2 & 6 \\ 0 & 1 & 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 & 1 & 7 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 & 6 \\ 0 & 1 & 1 & 2 & 4 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 4 & 1 & 7 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 & 1 & 7 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The resulting system is consistent, so $\mathbf{b}_1 \in H$. Setting the free variable $x_3 = 0$ gives the solution $x_4 = 1$, $x_2 = 2$ and $x_1 = 4$. Thus $\mathbf{b}_1 = 4\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4$ Note that there are three pivots so $\dim H = 3$ that correspond to the basis for H, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. The formula says can replace one of these vectors by $\mathbf{b}_1$ and it remains a generating set. Taking $\mathbf{b}_1$, $\mathbf{b}_2 = \mathbf{v}_1$ and $\mathbf{b}_3 = \mathbf{v}_2$, the generating set for H is

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 7 \\ 9 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Since the three vectors of $\mathcal{B}$ generate the space H of dimension three, $\mathcal{B}$ is a basis.

4. (a) *Let $\mathcal{B} = \{1, 2\sin^2 t\}$ and $\mathcal{C} = \{\sin^2 t, \cos^2 t\}$ be two bases of a vector space V. Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. If $[\mathbf{f}]_{\mathcal{C}} = \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]$, find $[\mathbf{f}]_{\mathcal{B}}$.*

We see that $\mathbf{b}_1 = 1 = \sin^2 t + \cos^2 t = \mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = 2\sin^2 t = 2\mathbf{c}_1$ so that the change of basis matrix is

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

The relation is

$$[\mathbf{f}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \, [\mathbf{f}]_{\mathcal{B}}$$

so

$$[\mathbf{f}]_{\mathcal{B}} = \left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \right)^{-1} [\mathbf{f}]_{\mathcal{C}} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -\frac{1}{2} \end{bmatrix}.$$

(b) *Find the row space of A, the null space of A and verify that* $\text{Row}(A) \perp \text{Nul}(A)$.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 3 & 6 & 4 \\ 4 & 8 & 6 \end{bmatrix}$$

Row reducing,

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ 3 & 6 & 4 \\ 4 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution of $A\mathbf{x} = \mathbf{0}$ is $x_3 = 0$ and $x_1 = -2x_2$ where x_2 is free and takes any value. Thus $\text{Row}(A) = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\text{Nul}(A) = \text{span}\{\mathbf{v}_1\}$ where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

The spaces are orthogonal if all pairs of basic vevtors are orthogonal. We see that $\mathbf{u}_1 \bullet \mathbf{v}_1 = 0$ and $\mathbf{u}_2 \bullet \mathbf{v}_1 = 0$ so $\text{Row}(A) \perp \text{Nul}(A)$.

5. *Consider two bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbf{R}^3$. Find $[\mathbf{w}]_\mathcal{B}$ and $[\mathbf{w}]_\mathcal{C}$. Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Check that your $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ changes $[\mathbf{w}]_\mathcal{B}$ to $[\mathbf{w}]_\mathcal{C}$.*

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}, \qquad C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Let

$$P_\mathcal{B} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \qquad P_\mathcal{C} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

We have $P_\mathcal{B}[\mathbf{w}]_\mathcal{B} = \mathbf{w}$ so

$$[\mathbf{w}]_\mathcal{B} = (P_\mathcal{B})^{-1}\,\mathbf{w} = \frac{1}{2\cdot 4 - 1\cdot 3}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix},$$

$$[\mathbf{w}]_\mathcal{C} = (P_\mathcal{C})^{-1}\,\mathbf{w} = \frac{1}{1\cdot 5 - 2\cdot 2}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = (P_\mathcal{C})^{-1}P_\mathcal{B}$ we reduce the augmented matrix

$$[P_\mathcal{C}|P_\mathcal{B}] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 5 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \mathit{to} \begin{bmatrix} 1 & 0 & 4 & -3 \\ 0 & 1 & -1 & 2 \end{bmatrix} = \left[I \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \right].$$

We find that the $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ does change $[\mathbf{w}]_\mathcal{B}$ to $[\mathbf{w}]_\mathcal{C}$, namely,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\mathbf{w}]_\mathcal{B} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = [\mathbf{w}]_\mathcal{C}.$$

6. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find matrices for which the statement fails.*

(a) STATEMENT. *If A is a 2×2 matrix with eigenvalue λ of multiplicity two then A is similar to* $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

FALSE. There may not be two independent eigenvectors corresponding to λ. An example of a matrix A with double eigenvalue $\lambda = 0$ which is not similar to D is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If similar we would have $A = PDP^{-1}$ for some invertible P. But this never holds because $PDP^{-1} = 0$ for all P.

(b) STATEMENT. $\mathcal{C}[0,1]$, *the vector space of continuous real valued functions on the unit interval is finite dimensional.*

FALSE. The vector space of functions $\mathcal{C}[0,1]$ contains the space of all polynomials $\mathbb{P}$. Any finite set $S \subset \mathcal{C}[0,1]$ with n elements cannot span $\mathcal{C}[0,1]$ because if it could, $\mathbb{P} \subset \operatorname{span}(S)$. But the subspace $H = \operatorname{span}\{1, t, t^2, \dots, t^n\} \subset \mathbb{P}$ of dimension $n+1$ cannot satisfy $H \subset \mathbb{P} \subset \operatorname{span}(S)$ because it does not satisfy the dimension inequality $\dim H \le \dim \operatorname{span}(S) \le n$ which would have to be true for finite dimensional subspaces since $H \subset \operatorname{span}(S)$.

(c) STATEMENT. *If A is a 2×2 real matrix and $\mathbf{b} \in \mathbf{R}^2$ is a vector such that $\mathbf{b} \perp \mathbf{w}$ for all $\mathbf{w}$ such that $A^T\mathbf{w} = \mathbf{0}$ then $A\mathbf{x} = \mathbf{b}$ is consistent.*

TRUE. The condition says that $\mathbf{b} \in \big(\operatorname{Nul}(A^T)\big)^{\perp} = \operatorname{Col}(A)$, so $A\mathbf{x} = \mathbf{b}$ is soluble.