Math 2270 § 4. **Third Midterm Part 1** Name: Solutions
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1. *Find x_2 using Cramer's Rule. Other methods will receive zero credit.*

$$\begin{aligned} x_1 + 4x_2 + \ x_3 &= \ \ 3 \\ 5x_1 + 3x_2 \qquad\ \ &= \ \ 9 \\ x_2 + 3x_3 &= -1 \end{aligned}$$

The equation is equivalent to $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 5 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 9 \\ -1 \end{bmatrix}$$

Cramer's rule for x_3 is

$$x_2 = \frac{\det(A_2(\mathbf{b}))}{\det(A)} = \frac{\begin{vmatrix} 1 & 3 & 1 \\ 5 & 9 & 0 \\ 0 & -1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 1 \\ 5 & 3 & 0 \\ 0 & 1 & 3 \end{vmatrix}} = \frac{27 - 5 - 45}{9 + 5 - 60} = \frac{-23}{-46} = \frac{1}{2}.$$

2. (a) *Let*

$$\mathcal{S} = \left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \\ 4 \end{bmatrix} \right\},$$

Let $\mathcal{H} = \operatorname{span}(\mathcal{S})$. Find a basis for $\mathcal{H}$. What is the dimension of $\mathcal{H}$?

Put the vectors as columns into a matrix. Then $\mathcal{H} = \operatorname{Col}(A)$. Do row reduction to identify pivot columns, which will form a basis of $\mathcal{H}$.

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 4 & 2 & 1 & 0 & 5 \\ 4 & 2 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus columns 1,3 and 4 are pivots. The basis for $\mathcal{H}$ is thus

$$\text{A basis for } \mathcal{H} \text{ is} \qquad \left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\},$$

(b) *Let*

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}, \qquad \mathbf{x} = \begin{bmatrix} 0 \\ 6 \\ 7 \\ 2 \end{bmatrix}.$$

Let $\mathcal{B}$ be a basis for the subspace $\mathcal{K} \subset \mathbf{R}^4$. Show $\mathbf{x} \in \mathcal{K}$ and find the coordinates $[\mathbf{x}]_{\mathcal{B}}$.

Put $\mathcal{B}$ and $\mathbf{y}$ in as columns of an augmented matrix. Row reduce to see if the system is consistent.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 6 \\ 0 & 2 & 1 & 7 \\ 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 6 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 7 \\ 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & -3 & 0 & -12 \\ 0 & 2 & 1 & 7 \\ 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 2 & 1 & 7 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, so there is a solution to $A\mathbf{c} = \mathbf{y}$. Solving, we find $c_3 = -1$, $c_2 = 4$ and $c_1 = 6 - 2c_2 = -2$. Thus, the coordinates are

$$[\mathbf{x}]_{\mathcal{B}} = \mathbf{c} = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$$

3. *Find the determinant* Δ *in two ways, using expansion by cofactors and using row operations.*

$$\begin{vmatrix} 2 & 1 & 0 & 3 \\ 1 & 3 & 0 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 2 & 3 & 2 \end{vmatrix}$$

Expanding by cofactors down the third column we find

$$\Delta = 1 \cdot \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 0 & 2 & 2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 1 & 0 & 3 \end{vmatrix} = (12 + 6 - 8 - 2) - 3(18 + 2 - 3 - 9) = 8 - 24 \cdot -16.$$

Using row operations we find

$$\Delta = \begin{vmatrix} 2 & 1 & 0 & 3 \\ 1 & 3 & 0 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 2 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 & 2 \\ 2 & 1 & 0 & 3 \\ 1 & 0 & 1 & 3 \\ 0 & 2 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & -5 & 0 & -1 \\ 0 & -3 & 1 & 1 \\ 0 & 2 & 3 & 2 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & -3 & 1 & 1 \\ 0 & 2 & 3 & 2 \end{vmatrix}$$
$$= 5 \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & \frac{8}{5} \\ 0 & 0 & 3 & \frac{8}{5} \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & \frac{8}{5} \\ 0 & 0 & 0 & \frac{8-24}{5} \end{vmatrix} = 5 \cdot \left(-\frac{16}{5}\right) = -16.$$

4. *Let $\mathcal{I} : \mathbb{P}_2 \to \mathbb{P}_3$ be the mapping given by definite integration $\mathcal{I}(\mathbf{p})(t) = \int_0^t \mathbf{p}(s)\,ds$, where $\mathbb{P}_3 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_0, a_1, a_2, a_3 \in \mathbf{R}\}$ is the vector space of polynomials of degree at most three. Show $\mathcal{I} : \mathbb{P}_2 \to \mathbb{P}_3$ is linear. Find the kernel (nullspace)* $\ker(\mathcal{I})$*. Find the range* $\mathrm{R}(\mathcal{I})$ *of* $\mathcal{I}$.

   To prove $\mathcal{I}$ is linear, we show that it preserves addition and multiplication by a constant. The fundamental reason is that integration is linear. For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}_2$ and $c \in \mathbf{R}$ we have

$$\begin{aligned}\mathcal{I}(\mathbf{p}+\mathbf{q})(t) &= \int_0^t (\mathbf{p}+\mathbf{q})(s)\,ds = \int_0^t \mathbf{p}(s) + \mathbf{q}(s)\,ds \\ &= \int_0^t \mathbf{p}(s)\,ds + \int_0^t \mathbf{q}(s)\,ds = \mathcal{I}(\mathbf{p})(t) + \mathcal{I}(\mathbf{q})(t) \\ \mathcal{I}(c\mathbf{p})(t) &= \int_0^t (c\mathbf{p})(s)\,ds = \int_0^t c\mathbf{p}(s)\,ds = c\int_0^t \mathbf{p}(s)\,ds = c\mathcal{I}(\mathbf{p})(t)\end{aligned}$$

   We could have worked out the map $\mathcal{I}$ on the coefficients of the polynomial, which would have resulted in an argument using linear formulas.

   Let us work out the map in terms of coefficients. For $\mathbf{p} \in \mathbb{P}_2$, there are constants a_0, a_1 and a_2 such that

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2.$$

   Thus

$$\mathcal{I}(\mathbf{p})(t) = \int_0^t \mathbf{p}(s)\,ds = a_0 t + \frac{1}{2}a_1 t^2 + \frac{1}{3}a_2 t^3.$$

   The kernel of $\mathcal{I}$ is the space of polynomials in $\mathbb{P}_2$ that are sent to zero.

$$\begin{aligned}\ker(\mathbf{I}) &= \{\mathbf{p} \in \mathbb{P}_2 : \mathcal{I}(\mathbf{p}) = \mathbf{0}\} = \left\{a_0 + a_1 t + a_2 t^2 : a_0 t + \frac{1}{2}a_1 t^2 + \frac{1}{3}a_2 t^3 = 0\right\} \\ &= \left\{a_0 + a_1 t + a_2 t^2 : a_0 = a_1 = a_2 = 0\right\} = \{\mathbf{0}\}\end{aligned}$$

   Thus the kernel is trivial.

   The range of $\mathcal{I}$ is the space of polynomials in $\mathbb{P}_3$ that are the images of vectors in $\mathbb{P}_2$.

$$\begin{aligned}\mathrm{R}(\mathbf{I}) &= \{\mathcal{I}(\mathbf{p}) : \mathbf{p} \in \mathbb{P}_2\} = \left\{a_0 t + \frac{1}{2}a_1 t^2 + \frac{1}{3}a_2 t^3 : a_0 + a_1 t + a_2 t^2 \in \mathbb{P}_2\right\} \\ &= \left\{b_0 t + b_1 t^2 + b_2 t^3 : b_0, b_1, b_2 \in \mathbf{R}\right\} = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_0 = 0\}\end{aligned}$$

   Thus the range is the three dimensional space $\{\mathbf{p} \in \mathbb{P}_3 : \mathbf{p}(0) = \mathbf{0}\}$. To see that every polynomial with zero constant coefficient is in $\mathrm{R}(T)$, choose any $\mathbf{q}(t) = b_0 t + b_1 t^2 + b_2 t^3$. Let

$$\mathbf{p}(t) = b_0 + 2b_1 t + 3b_2 t^2 \in \mathbb{P}_2.$$

   Then $\mathcal{I}(\mathbf{p}) = \mathbf{q}$.

5. (a) *Let* $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ *be an* $n \times n$ *matrix whose columns are* $\mathbf{a}_i$. *Let*

$$T(\mathbf{x}) = \det([\mathbf{a}_1\ \cdots \mathbf{a}_{n-1}\ \mathbf{x}])$$

be the function of the vector $\mathbf{x} \in \mathbf{R}^n$ *that evaluates the determinant of* A *with* $\mathbf{x}$ *replacing the last column. Show that* $T(\mathbf{x})$ *is a linear transformation.*

We have $T : \mathbf{R}^n \to \mathbf{R}$. The column expansion is linear in $\mathbf{x}$. To see this, expanding the determinant on the last column,

$$T(\mathbf{x}) = \sum_{k=1}^{n} b_k x_k, \qquad \text{where } b_k = (-1)^{k+n} \det(A_{k,n}),$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix gotten by crossing out the ith row and jth column of A. This way we see that T is a matrix transformation where M is the $1 \times n$ matrix

$$T(\mathbf{x}) = M\mathbf{x}, \qquad \text{where} \qquad M = [b_1, b_2, \ldots, b_n].$$

Hence T is linear. To verify, we may continue to check the two conditions for linearity. For any $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ and any $c \in \mathbf{R}$ we have

$$T(\mathbf{u}+\mathbf{v}) = M(\mathbf{u}+\mathbf{v}) = M\mathbf{u} + M\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}),$$
$$T(c\mathbf{u}) = M(c\mathbf{u}) = cM\mathbf{u} = cT(\mathbf{u}).$$

(b) *Find a linear transformation that takes the unit circle* $C = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$ *to the ellipse* $\mathcal{E}$ *that passes through the points* $(2,2)$, $(-1,1)$, $(-2,-2)$ *and* $(1,-1)$. *Use* T *to find the area inside* $\mathcal{E}$.

Let T be the transformation that takes the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ from the circle to the vectors $[2,2]$ and $[-1,1]$, resp. Thus

$$T(\mathbf{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = M\mathbf{x}.$$

and $T(\mathcal{C}) = \mathcal{E}$. Using the change of area formula

$$\text{Area}(\mathcal{E}) = |\det(M)|\ \text{Area}(\mathcal{C}) = \left| \det \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \right| \text{Area}(\mathcal{C}) = 4\pi.$$

(c) *Construct a* 4×4 *matrix* A *with* $\operatorname{rank}(A) = 2$. *Justify your construction.*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A has two pivots and rank is the number of pivots.

6. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find a counterexample for which the statement fails.*

(a) Statement. *Let* $\mathcal{M} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{R}\right\}$ *be the vector space of* 2×2 *matrices. Then* $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ *is the only basis for* $\mathcal{M}$.

False. There are many bases. Another one is $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

(b) STATEMENT. *For vectors* $\mathbf{u}, \mathbf{v}$ *in a vector space* $\mathbb{V}$*, if the equation* $\mathbf{u} + \mathbf{x} = \mathbf{v}$ *has a solution* $\mathbf{x} \in \mathbb{V}$ *then it is unique.*

TRUE. We assume there are two solutions $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{V}$ and show that they must be equal. A casual answer invoking the axioms would receive full credit. Here is a strict proof that $\mathbf{x} = \mathbf{y}$. Start from a known equation and replace by justifiably equal equations until you reach the conclusion.

$\mathbf{u} + \mathbf{x} = \mathbf{v}, \quad \mathbf{u} + \mathbf{y} = \mathbf{v}$	Both $\mathbf{x}$ and $\mathbf{y}$ satisfy the equation.
$\mathbf{u} + \mathbf{x} = \mathbf{u} + \mathbf{y}$	Both equal $\mathbf{v}$.
$(-\mathbf{u}) + (\mathbf{u} + \mathbf{x}) = (-\mathbf{u}) + (\mathbf{u} + \mathbf{y})$	V5. There is $-\mathbf{u}$. Pre-add to both sides.
$((-\mathbf{u}) + \mathbf{u}) + \mathbf{x} = ((-\mathbf{u}) + \mathbf{u}) + \mathbf{y}$	V3. Associativity of addition.
$(\mathbf{u} + (-\mathbf{u})) + \mathbf{x} = (\mathbf{u} + (-\mathbf{u}) + \mathbf{y}$	V2. Commutativity of addition.
$\mathbf{0} + \mathbf{x} = \mathbf{0} + \mathbf{x}$	V5. Additive inverse.
$\mathbf{x} + \mathbf{0} = \mathbf{y} + \mathbf{0}$	V1. Commutativity of addition.
$\mathbf{x} = \mathbf{y}$	V4. Additive Identity.

Another line of argument is since we are given that a solution exists we deduce that it must equal $\mathbf{x} = (-\mathbf{u}) + \mathbf{v}$, which is the same for every solution.

(c) STATEMENT. *Let* $\mathcal{C}[0,1] = \{f : [0,1] \to \mathbf{R} : f$ *is continuous*$\}$ *be the vector space of continuous functions on* $[0,1]$*. Then the subset* $\mathcal{H} = \{g \in \mathcal{C}[0,1] : g(1) = 0\}$ *is a subspace of* $\mathcal{C}[0,1]$*.*

We check the tree conditions to be a subspace.

(a) Zero is in $\mathcal{H}$: the zero in $\mathcal{C}[0,1]$ is the zero function $z(t) = 0$ for all t. $z(0) = 0$ so $z \in \mathcal{H}$.

(b) $\mathcal{H}$ is closed under addition. Choose any $f, g \in \mathcal{H}$ so $f(1) = g(1) = 0$. But then the sum $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$ so $f + g \in \mathcal{H}$.

(c) $\mathcal{H}$ is closed under multiplication by a scalar. Choose any $f \in \mathcal{H}$ and $c \in \mathbf{R}$ so $f(1) = 0$. But then the multiple $(cf)(1) = cf(1) = c \cdot 0 = 0$ so $cf \in \mathcal{H}$.