Math 1310 Final Exam Practice Problems

1. (1.3) Consider the function

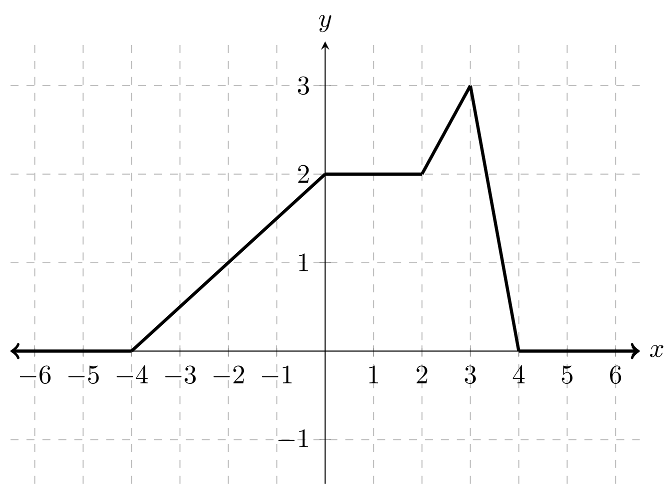
$$f(x) = \begin{cases} x+1, & x < 0 \\ 1-x^2, & x \in [0,1] \\ x-1, & x > 1 \end{cases},$$

and the transformed function $g(x) = -2f(-2x+3)$. Find the value of $g(5/4) = y$.

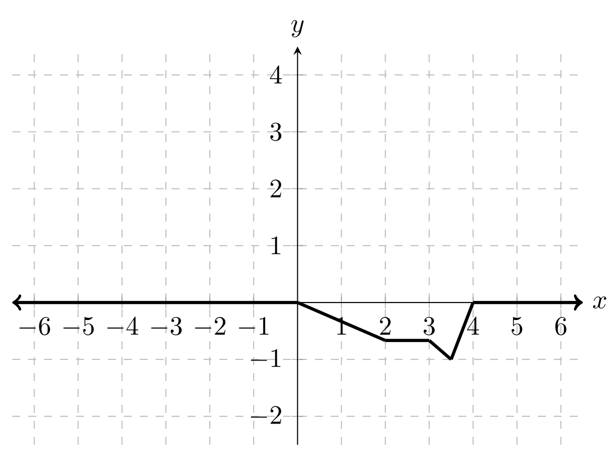
Solution: $g(\frac{5}{4}) = -2f(-\frac{5}{2}+\frac{6}{2}) = -2f(\frac{1}{2}) = -2[1-(\frac{1}{2})^2] = -2[\frac{4}{4}-\frac{1}{4}] = -\frac{3}{2}$.

2. (1.3) Consider the function $f(x)$ depicted in the graph. Draw a graph of the transformed function

$$-\frac{1}{3}f(2x-4).$$



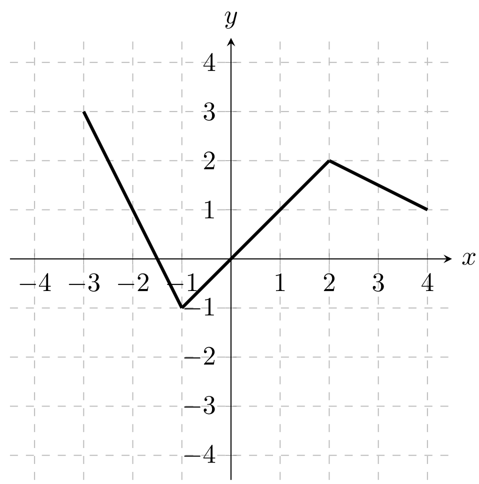
Solution: Its useful to express $-\frac{1}{3}f(2x-4) = -\frac{1}{3}f(2(x-2))$.



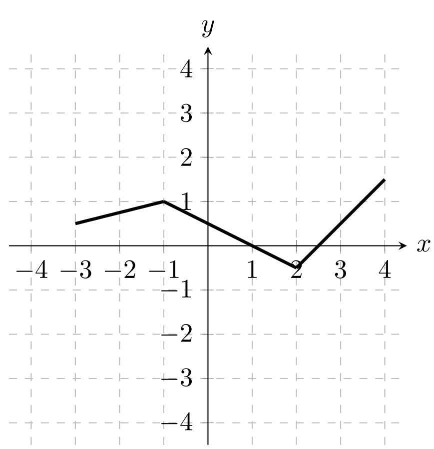
3. (1.3) Consider a function $f(x)$ with domain $(-4, 12]$ and range $[3, 7)$. What are domain and range of $f(-3x+1)+2$?

> **Solution:** Since the domain of $f(x)$ is $(-4, 12]$, the domain of $g(x) = f(-3x+1)+2$ consists of all x, such that
>
> $$-4 < -3x+1 \leq 12 \qquad -5 < -3x \leq 11 \qquad \frac{5}{3} > x \geq -\frac{11}{3}$$
>
> So the domain of $g(x)$ is $[-\frac{11}{3}, \frac{5}{3})$. The range is $[5, 9)$.

4. (1.3) Consider the given graph of $g(x) = 2f(-x+1)$. Draw the graph of $f(x)$.



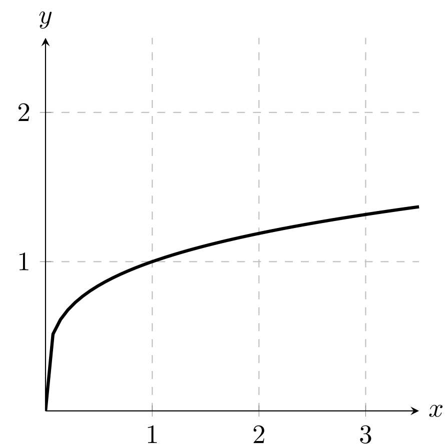
> **Solution:**
>
> $$\frac{1}{2}g(x) = f(-x+1) \qquad z = -x+1 \quad x = -z+1 \qquad f(x) = \frac{1}{2}g(-x+1)$$
>
> So the graph of $f(x)$ is



5. (1.3) Suppose that $f(x) = \sqrt{x}$ and $g(x) = \ln(x-1)$. Identify and sketch each of the function compositions below. What is the domain and range of each function composition?

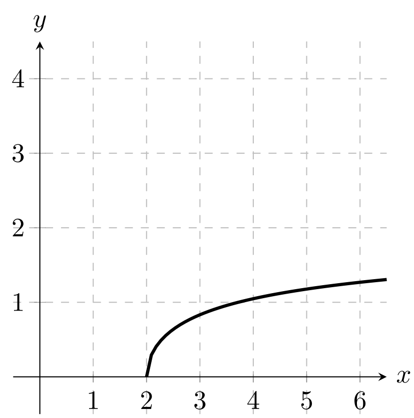
(a) $f \circ f$

Solution: $\big(f \circ f\big)(x) = x^{\frac{1}{4}}$. The domain is $[0, \infty)$ and the range is $[0, \infty)$.



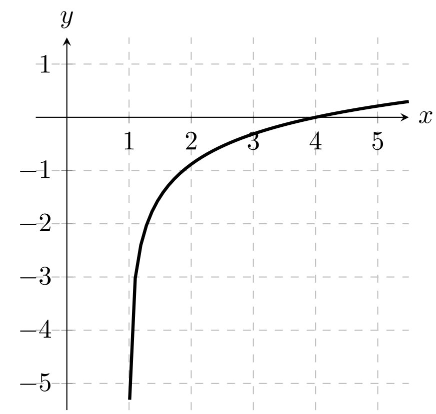
(b) $f \circ g$

Solution: $\big(f \circ g\big)(x) = \big(\ln(x-1)\big)^{\frac{1}{2}}$. The domain is $[2, \infty)$. The range is $[0, \infty)$.



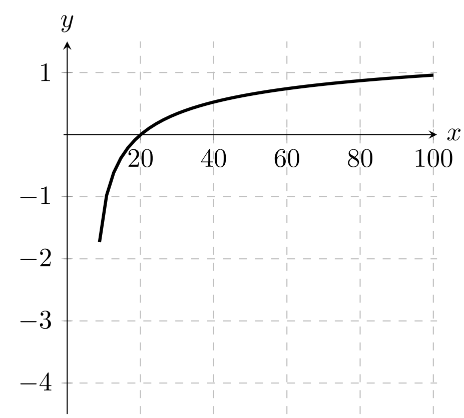
(c) $g \circ f$

Solution: $\big(g \circ f\big)(x) = \ln\big(\sqrt{x} - 1\big)$. The domain is $(1, \infty)$ and the range is $(-\infty, \infty)$.



(d) $g \circ g$

Solution: $(g \circ g)(x) = \ln\big((\ln(x) - 1) - 1\big)$. The domain is (e^2, ∞) and the range is $(-\infty, \infty)$.



6. (1.5) Find the domain of each function.

(a) $\dfrac{a^x - 4}{5^x - 25}$ with $a > 0$

Solution: $a^x - 4$ and $5^x - 25$ are defined everywhere, so we only require $5^x - 25 \neq 0$. Hence $x \neq 2$.

(b) $\ln(\cos(x) + 5)$

Solution: $\ln(x)$ is defined when $x > 0$, so we require $\cos(x) + 5 > 0$ which is always true. Hence the domain is $\mathbb{R}$.

(c) $\dfrac{e^{\sqrt{3^x - 27}}}{\sqrt{x^3 - 8}}$

Solution: Here we require $\sqrt{x}$ to be defined, and hence $3^x - 27 > 0$ and $x^3 - 8 > 0$. This implies $x > 3$. This also implies $\sqrt{x^3 - 8} > 0$, so the domain is $x > 3$.

7. (1.5) A population of shrimp is known to triple every year. Let $p(t)$ represent the population of shrimp after t years and suppose $p(0) = 100$.

(a) What is the equation for $p(t)$ as an exponential function?

Solution: Since the population triples,

$$\begin{aligned} p(1) &= 3p(0) \\ p(2) &= 3p(1) = 3 \cdot 3p(0) \\ p(3) &= 3p(2) = 3 \cdot 3 \cdot 3p(0) \\ &\dots \\ p(t) &= 3^t p(0) = 100 \cdot 3^t \end{aligned}$$

(b) After how many years are there 900 shrimp?

Solution: At $t = 2$, $p(t) = 900$, so two years.

8. (1.6) Consider the discrete function $f(x)$ given by

$$f(1) = -5 \qquad f(2) = 0 \qquad f(3) = 7 \qquad f(4) = 2$$

(a) Explain, why $f(x)$ has an inverse function. What are domain and range of $f^{-1}(x)$?

Solution: No y-value is taken more then once, so $f(x)$ is one-to-one and $f(x)$ has an inverse. The domain of $f^{-1}(x)$ is $\{-5, 0, 2, 7\}$ and the range $\{1, 2, 3, 4\}$.

(b) What is $f^{-1}(2)$?

Solution:

$$f^{-1}(2) = 4$$

(c) What are $f(f^{-1}(2))$ and $f^{-1}(f(2))$?

Solution: Both are 2.

9. (1.6) Determine, whether $f(x) = (\sin(x))^3$ has an inverse for the given domains. If the inverse function exists, what is its domain and range?

(a) $(0, \pi)$

Solution: $f(x)$ has no inverse on this domain, since $f(\frac{\pi}{4}) = f(\frac{3\pi}{4})$.

(b) $(0, \frac{\pi}{2})$

Solution: $f(x)$ has an inverse on this domain, since $f(x)$ passes the horizontal line test, it is an increasing function from 0 to 1. So f^{-1} has domain $(0, 1)$ and range $(0, \frac{\pi}{2})$.

(c) $(-\frac{\pi}{2}, \frac{\pi}{2})$

Solution: $f(x)$ has an inverse on this domain, since $f(x)$ passes the horizontal line test, it is an increasing function from -1 to 1. So f^{-1} has domain $(-1, 1)$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$.

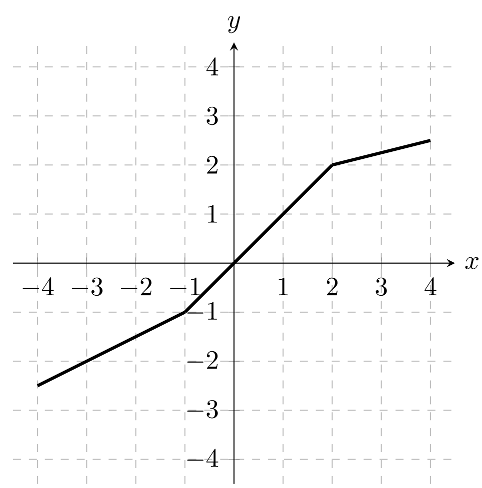
10. (1.6) Consider the function $f(x) = x^2 + 2x + 1$ on the domain $(-\infty, -1)$. Find the inverse function $f^{-1}(x)$. What is its domain and range?

Solution:

$$y = x^2 + 2x + 1 = (x+1)^2 \qquad \sqrt{y} = \pm(x+1) \qquad x = -1 \pm \sqrt{y}$$

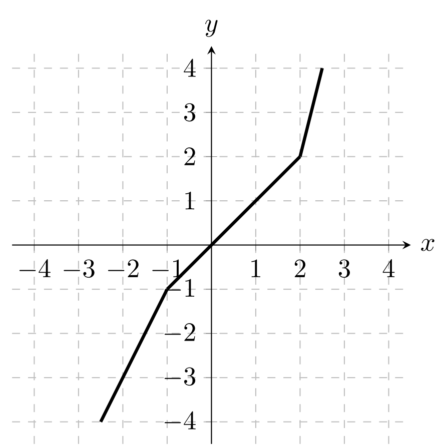
Since the range of $f^{-1}(x)$ is $(-\infty, -1)$, choose the negative sign, that is $f^{-1}(x) = -1 - \sqrt{x}$ with domain $(0, \infty)$ and range $(-\infty, -1)$.

11. (1.6) Consider the function $f(x)$ given by



Explain, why $f(x)$ has an inverse function. Draw the graph of $f^{-1}(x)$.

Solution: $f(x)$ passes the horizontal line test, that is horizontal line crosses $y = f(x)$ at most once. The graph of $f^{-1}(x)$ is obtained, by reflecting $y = f(x)$ about $y = x$.



12. (1.6) Solve the equation for x:

$$10 = 2^{3x} e^{-x}.$$

Solution: We will use the natural log to solve, but any base is fine.

$$\ln(10) = \ln(2^{3x}e^{-x}) = 3x\ln(2) - x$$

so,

$$x = \frac{\ln(10)}{3\ln(2) - 1}.$$

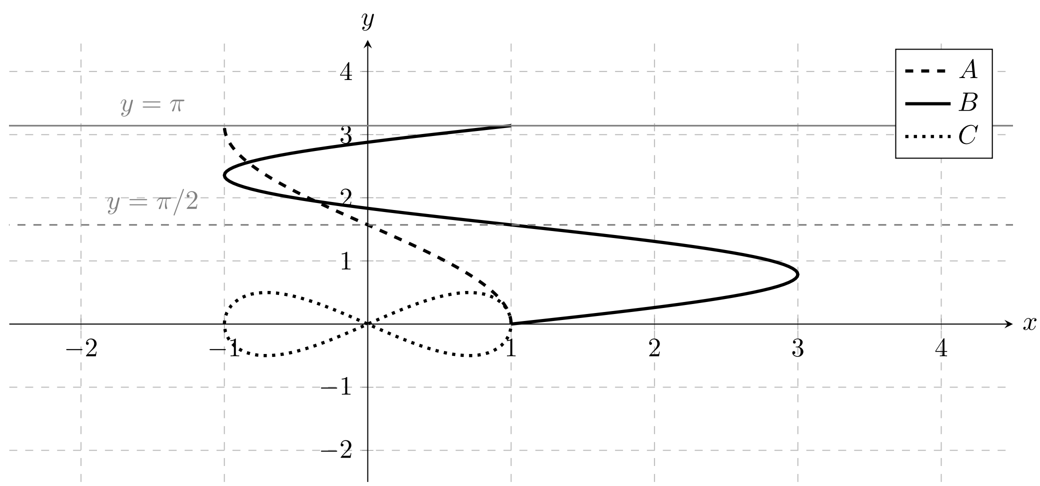
13. (1.7)

(a) Consider the parametric curve given by

$$x = 2\sin(2t) + 1, \quad y = t$$

for $0 \le t \le \pi$. Which curve in the figure below corresponds to these equations?

Solution: Curve B. This can be found by plotting a few points, but actually you only need to plot $t = \pi/2$ or $t = \pi$.



(b) Is there a function $y = f(x)$ that describes this curve?

Solution: No. Remember, a function takes each x value in its domain and gives a *unique* y value, but in the curve above, x values correspond to multiple y values.

14. (2.1) Find an equation of a secant line P on the curve defined by the function $f(x) = x^3 - x$, through the points $x = 1/2$ and $x = 1$.

Solution: The equation for a line is $y = mx + b$. It must pass through the points (1/2, - 3/8) and (1,0) The slope of the secant line is computed to be

$$m = \frac{f(1) - f(1/2)}{1 - 1/2} = \frac{0 - \frac{1}{8} + \frac{1}{2}}{\frac{1}{2}} = \frac{3}{4}.$$

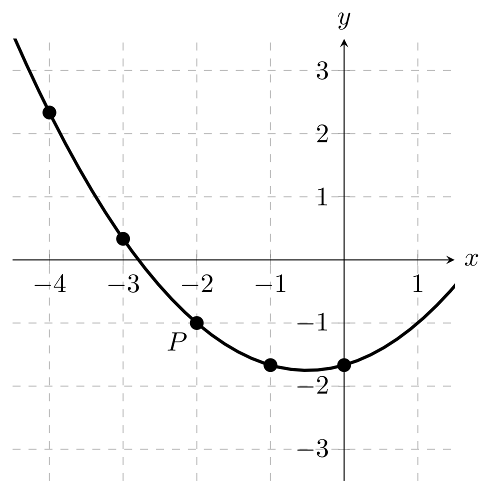
The y-intercept b is found by substituting one of the points. Let's use $(1, 0)$:

$$y = 0 = \frac{3}{4}(1) + b \implies b = -\frac{3}{4}.$$

The secant line is

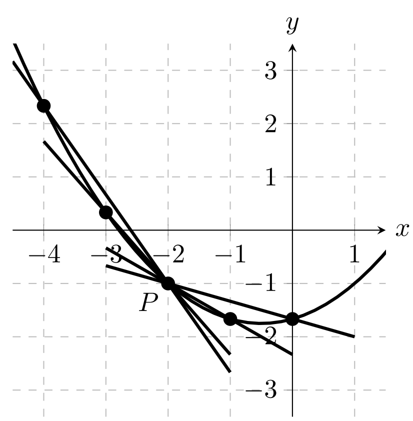
$$y = \frac{3}{4}x - \frac{3}{4}.$$

15. (2.1) Consider the function $f(x)$ given by the graph and the point P.



(a) Draw the secant lines $y = mx + b$ through P and each of the other given points.

Solution:



(b) What is the slope m of the secant lines?

Solution:

$x = -4$**:** $m = -\frac{5}{3}$

$x = -3$**:** $m = -\frac{4}{3}$

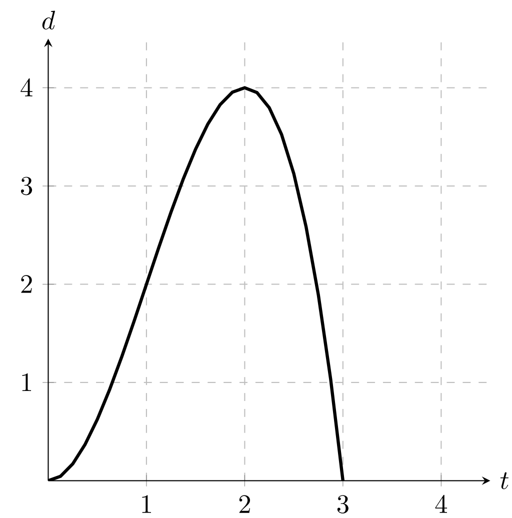
$x = -1$**:** $m = -\frac{2}{3}$

$x = 0$**:** $m = -\frac{1}{3}$

(c) Estimate the slope of the tangent line at P.

Solution: The slope of the tangent line should be between $-\frac{4}{3}$ and $-\frac{2}{3}$. So -1 is a good guess.

16. (2.1) The distance of an object from its starting point satisfies the equation $d(t) = 3t^2 - t^3$, and is graphed below.



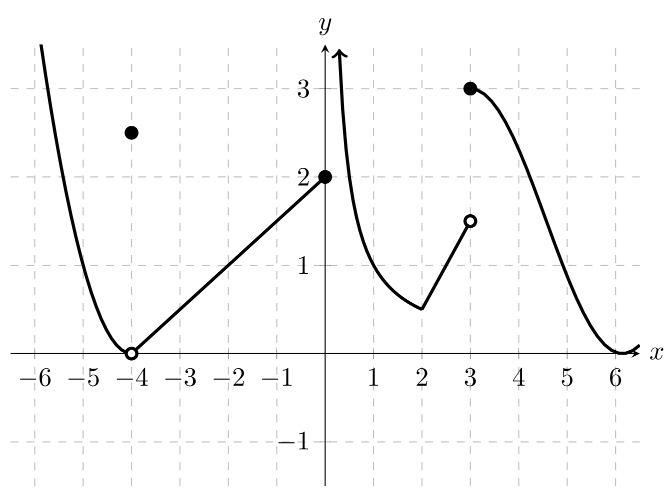
(a) What is the average velocity over the following intervals, $t \in [1, 2], t \in [1, 1.1], t \in [1, 1.01]$, correct to two decimal places?

Solution: The average velocities are respectively 2, 2.99 and 3.00.

(b) What is the approximate slope of the tangent to the curve at $t = 1$?

Solution: The gradient is 3, as this is what the average velocities are converging to.

17. (2.2) The graph below depicts a function $f(x)$. Compute the following limits using the graph if the limit exists. If the limit exists, specify why, and specify if it is infinite where appropriate.



(a) $\lim_{x\to -4^-} f(x) = 0$

(b) $\lim_{x\to -4^+} f(x) = 0$

(c) $\lim_{x\to -4} f(x) = 0$

(d) $\lim_{x\to 0^-} f(x) = 2$

(e) $\lim_{x\to 0^+} f(x) = \infty$

(f) $\lim_{x\to 0} f(x)$ DNE because left and right limits do not agree

(g) $\lim_{x\to 3^-} f(x) = 1.5$

(h) $\lim_{x\to 3^+} f(x) = 3$

(i) $\lim_{x\to 3} f(x)$ DNE because left and right limits do not agree

18. (2.3) Consider functions $f(x)$, $g(x)$ and $h(x)$ with

$$\lim_{x\to 3} f(x) = 4 \qquad \lim_{x\to 3} g(x) = 12 \qquad \lim_{x\to 3} h(x) = -2$$

Calculate the following limit.

$$\lim_{x\to 3} \frac{f(x) - g(x)}{g(x)h(x)}$$

Solution:

$$\lim_{x\to 3} \frac{f(x) - g(x)}{g(x)h(x)} = \frac{\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)}{(\lim_{x\to 3} g(x))(\lim_{x\to 3} h(x))} = \frac{4 - 12}{12 \cdot (-2)} = \frac{1}{3}.$$

19. (2.3) Find the following limit

$$\lim_{x\to 2} \frac{x^2 - \sqrt{x+1}}{x^2 + 1}$$

Solution: 2 lies in the domain of $\frac{x^2-\sqrt{x+1}}{x^2+1}$. So by the direct substitution property, when taking the limit, it is enough to substitute x by 2, that is

$$\lim_{x\to 2} \frac{x^2 - \sqrt{x+1}}{x^2 + 1} = \frac{2^2 - \sqrt{2+1}}{2^2 + 1} = \frac{4 - \sqrt{3}}{5}$$

20. (2.3) Find the following limit

$$\lim_{x\to 6} \frac{x^2 - 36}{x^2 - 4x - 12}$$

Solution: The limit of the numerator and denominator as x approaches 6 are both 0. By factoring, one gets

$$\lim_{x\to 6} \frac{x^2 - 36}{x^2 - 4x - 12} = \lim_{x\to 6} \frac{(x-6)(x+6)}{(x-6)(x+2)} = \lim_{x\to 6} \frac{x+6}{x+2} = \frac{12}{8} = \frac{3}{2}$$

21. (2.3) Consider the function $f(x) = e^x + x^2 \sin\left(\frac{1}{x}\right)$.

(a) Find functions $g(x)$ and $h(x)$, such that $g(x) \le f(x) \le h(x)$. Where does the inequality hold?

Solution: It is

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$e^x - x^2 \leq f(x) \leq e^x + x^2$$

So choose $g(x) = e^x - x^2$ and $h(x) = e^x + x^2$. The inequality holds for all real numbers.

(b) Find $\lim_{x \to 0} f(x)$

Solution: It is

$$\lim_{x \to 0} g(x) = 1 \qquad \lim_{x \to 0} h(x) = 1$$

So the squeeze theorem implies

$$\lim_{x \to 0} f(x) = 1$$

22. (2.4) Determine if the function $f(x) = \frac{x^2-4x+3}{x^2-1}$ is continuous at the following points:

(a) $x = 2$

Solution: The function $f(x)$ is a rational function and the input $x = 2$ is within the domain of $f(x)$, therefore it is continuous there.

(b) $x = 1$

Solution: The limit $\lim_{x \to 1} \frac{x^2-4x+3}{x^2-1} = \lim_{x \to 1} \frac{(x-1)(x-3)}{(x-1)(x+1)} = -1$ exists, however the function value $f(1)$ is not defined, therefore the function is not continuous at $x = 1$.

23. (2.4) The Heaviside step function $H(x)$ is defined as

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Determine if the function $f(x) = H(x) + H(-x)$ is continuous at $x = 0$.

Solution: The function is defined at $x = 0$ and is equal to $f(0) = H(0) + H(-0) = 1 + 1 = 2$.
Because the function is defined piecewise on domains $x < 0$ and $x \geq 0$, we must examine directional limits to determine if the two-sided limit exists:
Left limit: $\lim_{x \to 0^-}(H(x) + H(-x)) = 0 + 1 = 1$.
Right limit: $\lim_{x \to 0^+}(H(x) + H(-x)) = 1 + 0 = 1$.
Therefore $\lim_{x \to 0}(H(x) + H(-x)) = 1$ exists. However, $f(0) = 2 \neq 1$, so $f(x)$ is not continuous at $x = 0$.

24. (2.4) Consider the function

$$f(x) = \begin{cases} 1 - x^2, & x > 0 \\ x + 1, & x \leq 0 \end{cases}.$$

Identify all x-values on which $f(x)$ is continuous.

Solution: The piecewise defined function's two sub-functions are polynomials, so they are continuous on their entire domains. We need to check the single point $x = 0$ on the shared boundary of their two domains:

$$f(0) = 1$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1-x^2) = 1.$$

Therefore the two-sided limit exists: $\lim_{x\to 0} f(x) = 1$, and it agrees with the function's value there $f(0) = 1$, so the function is continuous for all real values x.

25. (2.4) Find all points of discontinuity for the function

$$f(x) = \frac{x^3 - x}{x^3 - 2x^2 + x}.$$

Solution: The denominator: $x^3 - 2x^2 + x = x(x-1)(x-1)$, so the points $x = 0, 1$ are undefined and so the function cannot be continuous there.

26. (2.4) Suppose a function $f(x)$ is continuous at $x = -2$, and $f(-2) = 4$. Determine the limit

$$\lim_{x\to -2} f(x)$$

Solution: Continuity of $f(x)$ at $x = -2$ means that the limiting value of $f(x)$ as $x \to -2$ is equal to the function's value at $x = -2$. So,

$$\lim_{x\to -2} f(x) = f(-2) = 4.$$

27. (2.4) Determine if $f(x) = e^x \sin(x)$ is continuous on the real line.

Solution: Both exponential functions and trigonometric functions are continuous on their domains; the domains of e^x and $\sin(x)$ are the real line. The product of any two continuous functions is continuous, therefore $e^x \sin(x)$ is continuous.

28. (2.4) Consider the function $f(x) = 2^{-x} - 2^{-2x}$ on the interval $x \in [0, 1]$. Determine if you can guarantee a solution to the equation $f(x) = \frac{1}{10}$ exists within $[0, 1]$ without solving the equation for x (note, you can't solve it explicitly anyway).

Solution: The function $f(x)$ is a sum of exponential functions and exponential functions are continuous everywhere so they are continuous on $[0, 1]$. Note also that $f(0) = 0$, and

$$f(1) = 2^{-1} - 2^{-2} = \tfrac{1}{2} - \tfrac{1}{2^2} = \tfrac{1}{4}.$$

It is true that $0 = f(0) < \frac{1}{10} < \frac{1}{4} = f(1)$.
The hypotheses of the intermediate value theorem are satisfied, so it is guaranteed there is an x within $[0, 1]$ that solves $f(x) = \frac{1}{10}$, even if it cannot be found explicitly.

29. (2.4)

(a) Use the Intermediate Value Theorem to show that $f(x) = 2x^3 - 5x + 1$ has a root between 1 and 2.

Solution: $f(1) = -2$ and $f(2) = 7$. Hence by the intermediate value theorem, there exists a number $a \in (1, 2)$ such that $f(a) = 0$.

(b) For what value of c is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} x^5 + cx^2 + 2 & \text{if } x < 2 \\ x^2 - cx & \text{if } x \geq 2. \end{cases}$$

Solution: For it to be continuous, the left hand limit must equal the right hand limit at $x = 2$. We thus require that

$$2^5 + c \times 2^2 + 2 = 2^2 - 2c.$$

Simplifying this equation, we find that $-6c = 30$, and therefore $c = -5$.

30. (2.5) Let $f(x) = \frac{1}{x^2 - \pi^2}$. Calculate the following limits with the appropriate infinite behavior, or specify that it does not exist. If it does not exist, specify why.

(a) $\lim\limits_{x \to \pi^-} f(x)$

Solution: $\lim\limits_{x \to \pi^-} f(x) = -\infty$

(b) $\lim\limits_{x \to \pi^+} f(x)$

Solution: $\lim\limits_{x \to \pi^+} f(x) = \infty$

(c) $\lim\limits_{x \to \pi} f(x)$

Solution: $\lim\limits_{x \to \pi} f(x)$ does not exist because the left- and right-directional limits do not agree.

31. (2.5) Let $f(x) = \frac{1}{(x^2 - 2)^2}$. Calculate the following limits with the appropriate infinite behavior, or specify that it does not exist. If it does not exist, specify why.

(a) $\lim\limits_{x \to \sqrt{2}^-} f(x)$

Solution: $\lim_{x \to \sqrt{2}^-} f(x) = \infty$

(b) $\lim_{x \to \sqrt{2}^+} f(x)$

Solution: $\lim_{x \to \sqrt{2}^+} f(x) = \infty$

(c) $\lim_{x \to \sqrt{2}} f(x)$

Solution: $\lim_{x \to \sqrt{2}} f(x) = \infty$, because left- and right-limits agree.

32. (2.5) Calculate the limit

$$\lim_{x \to -\infty} \frac{3x^3 + x^2 - 2x}{2x^3 - x + 1}$$

by algebraically manipulating the numerator and denominator and using limit laws but without using L'Hospital's rule.

Solution: Multiply by the factor $\frac{(\frac{1}{x^3})}{(\frac{1}{x^3})}$ and simplify

$$\lim_{x \to -\infty} \frac{3x^3 + x^2 - 2x}{2x^3 - x + 1} \frac{(\frac{1}{x^3})}{(\frac{1}{x^3})} = \lim_{x \to -\infty} \frac{3 + \frac{1}{x} - 2\frac{1}{x^2}}{2 - \frac{1}{x^2} + \frac{1}{x^3}}.$$

Each limit of each of the terms in the numerator and denominator exist, and the denominator does not approach zero, so the limit laws can be used

$$= \frac{\lim_{x \to -\infty} 3 + \lim_{x \to -\infty} \frac{1}{x} - \lim_{x \to -\infty} 2\frac{1}{x^2}}{\lim_{x \to -\infty} 2 - \lim_{x \to -\infty} \frac{1}{x^2} + \lim_{x \to -\infty} \frac{1}{x^3}} = \frac{3 + 0 - 0}{2 - 0 + 0} = \frac{3}{2}.$$

33. (2.5) Calculate the limit

$$\lim_{x \to -\infty} \frac{e^x}{x}$$

Solution: The product of $\frac{1}{x}$ and e^x form the expression, and the limits of each exist:

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to -\infty} \frac{1}{x} = 0.$$

Use the limit law: the limit of the product is the product of the limits

$$\lim_{x \to -\infty} \frac{e^x}{x} = \left(\lim_{x \to -\infty} e^x\right) \left(\lim_{x \to -\infty} \frac{1}{x}\right) = 0 \times 0 = 0.$$

34. (2.6) Consider the function $f(x) = \sqrt{x+1}$

(a) Compute the derivative of $f(x)$ at $a = 3$ using the limit-based definition

Solution:

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\sqrt{3+h+1} - \sqrt{3+1}}{h}$$
$$= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$$

(b) Find the tangent line of $f(x)$ at $a = 3$.

Solution:

$$y - 2 = \frac{1}{4}(x-3) \qquad y = \frac{1}{4}x + \frac{5}{4}$$

35. (2.6) A particle is moving in a straight line given by the equation $x(t) = \frac{1}{1+t}$, where t is measured in seconds and x is measured in meters. What is the velocity at time $t = 1\,\mathrm{s}$?

Solution:

$$v(1) = x'(1) = \lim_{h\to 0} \frac{x(1+h) - x(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+1+h} - \frac{1}{1+1}}{h}$$
$$= \lim_{h\to 0} \frac{2-(2+h)}{h2(2+h)} = \lim_{h\to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}\frac{\mathrm{m}}{\mathrm{s}}$$

36. (2.6) Consider a hot cup of tea. Let $T(t)$ be the temperature in Fahrenheit of a cup of tea at time t in minutes.

(a) What does $T'(10)$ represent? Is it positive, negative or zero?

Solution: $T'(10)$ is the rate of change, that is the rate in which the temperature changes after 10 minutes. It is negative, since the tea is cooling down.

(b) What is the unit of $T'(10)$?

Solution: Fahrenheit per minute

37. (2.6) Consider $f(x) = |x|$. Determine whether $f'(0)$ exists using the limit definition of the derivative.

Solution: $f'(0)$ does not exist, since

$$\lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$
$$\lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

38. (2.7) Compute the derivative of $f(x) = x^3 - 2x$ as a function of x using the limit definition.

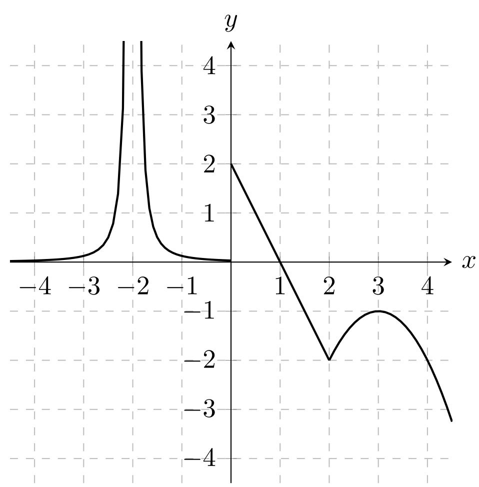
Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - 2(x+h) - x^3 + 2x}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x - 2h - x^3 + 2x}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 2) = 3x^2 - 2 \end{aligned}$$

39. (2.7) Circle all expressions that represent the first derivative of $y = f(x)$.

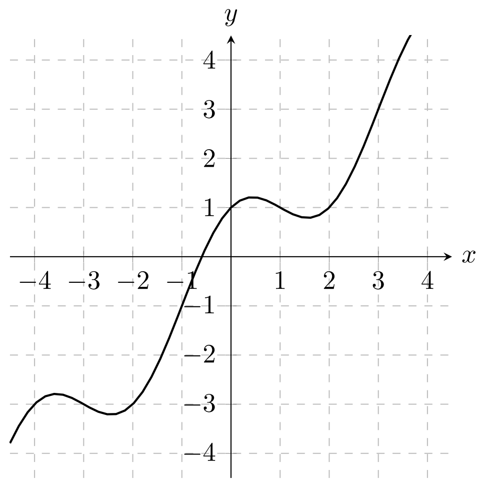
○ d′	√ $f'(x)$	√ y'
√ $\mathrm{D}_x f(x)$	○ $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$	○ $f'(y)$

40. (2.7) Consider the graph of $f(x)$. Find all the x-values, where the derivative of $f(x)$ does not exist. Explain, why it does not exist. If it exists everywhere, explain why.

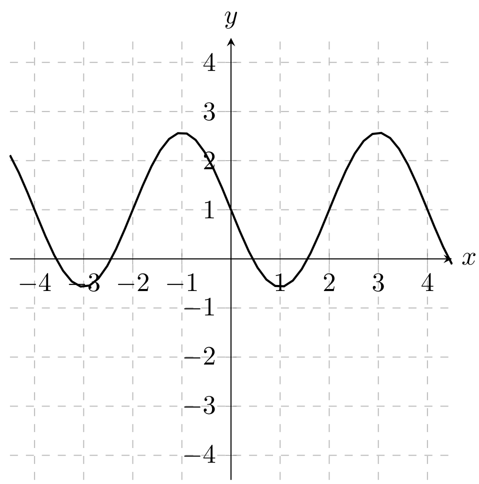


Solution: The derivative does not exist at $x = -2$, because the function is not defined there. It does not exist at $x = 0$, because the function has a jump there and is not even continuous. It does not exist at $x = 2$, because the function has a corner and the graph has no tangent line there.

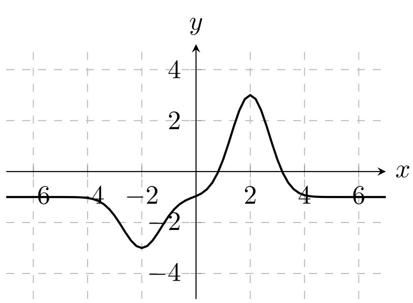
41. (2.8) Consider the function $f(x)$ given by the following graph.



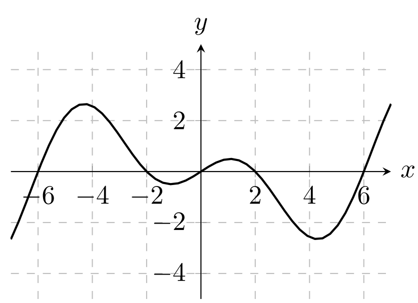
Sketch the derivative of $f(x)$. Make sure to identify, where $f'(x)$ is positive and where it is negative.



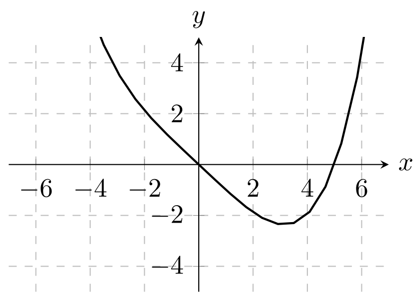
42. (2.8) Match the graphs of the four functions ((A)–(D)) given with the graphs of their derivatives ((a)–(d)). Explain your answer. For example, identify some distinct feature(s) of the graphs.



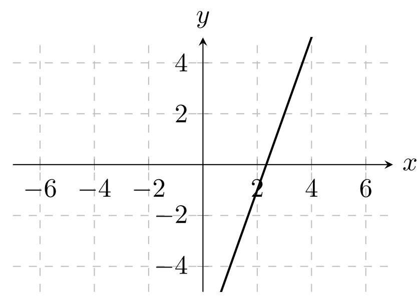
(A)



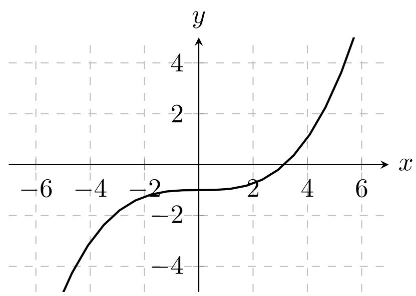
(B)



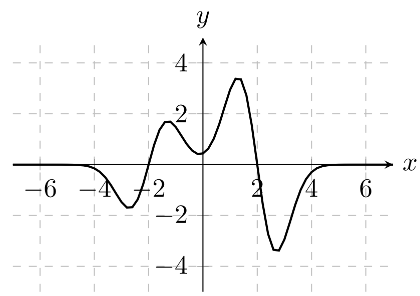
(C)



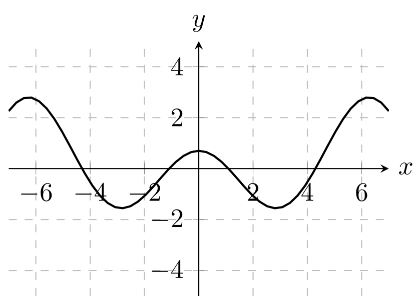
(D)



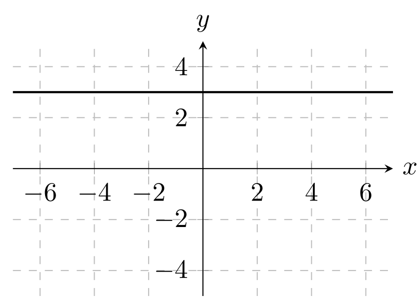
(a)



(c)



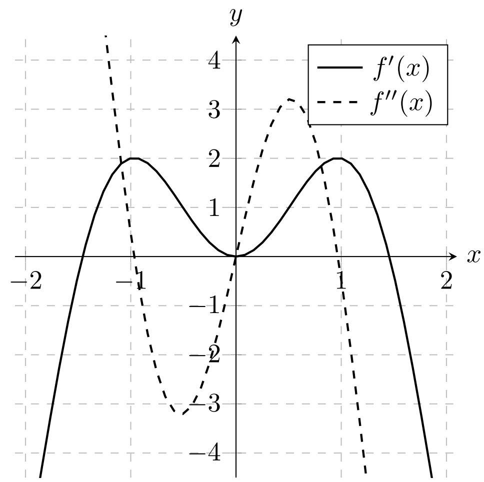
(b)



(d)

Solution: (D) has a constant slope, so it corresponds to (d)
(A) is constant for $x < -5$ and $x > 5$, so it corresponds to (c)
(B) has a point symmetry, so the derivative will have a symmetry about the y-axis and it corresponds to (b)
(C) decreases until $x \approx 3$ and then it increases, so it corresponds to (a)

43. (2.8) Consider the graphs of $f'(x)$ and $f''(x)$



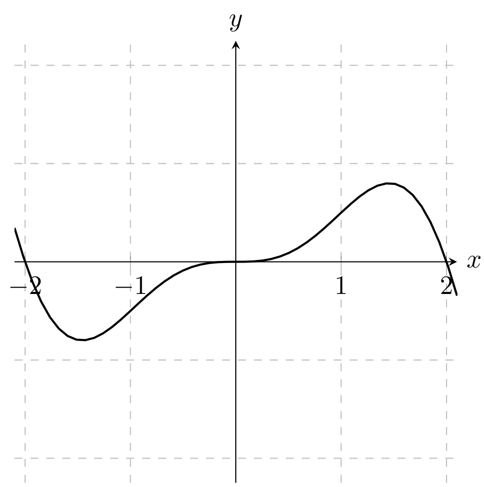
(a) Where is $f(x)$ increasing? Where is it decreasing?

Solution: f is increasing for $-1.5 < x < 0$ and $0 < x < 1.5$. f is decreasing for $x < -1.5$ and for $x > 1.5$.

(b) Where is $f(x)$ concave up? Where is it concave down?

Solution: f is concave up for $x < -1$ and $0 < x < 1$. f is concave down for $-1 < x < 0$ and $x > 1$.

(c) Sketch a possible graph of $f(x)$.



44. (2.8) Consider the function $f(x) = x^3 - 3x - 1$. The first two derivatives are $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$.

(a) Find all the zeros of f' and f''.

Solution:

$$f'(x) = 3(x^2 - 1) = 3(x - 1)(x + 1)$$

So the zeros of f' are 1, -1 and the zero of f'' is 0.

(b) Where are the maxima and minima of f?

Solution: f has a maximum at -1, since f comes from $-\infty$ from the left and a minimum at 1.

(c) Where are the inflection points of f?

Solution: f has an inflection point at 0.

45. (3.1) Suppose f and g are differentiable and $f'(2) = 4$ and $g'(2) = -3$. Compute the derivative of $h(x) = 2f(x) - 4g(x)$ at $x = 2$.

Solution:
$$h'(x) = 2f'(2) - 4g'(x) = 2(4) - 4(-3) = 8 + 12 = 20$$

46. (3.1) Compute $\lim_{h\to 0} \frac{e^h-1}{h}$

Solution: By definition
$$\lim_{h\to 0} \frac{e^h-1}{h} e^x = e^x$$
So,
$$\lim_{h\to 0} \frac{e^h-1}{h} = 1$$

47. (3.1) Compute the derivatives of the following functions

(a) x^{11}

Solution: $11x^{10}$

(b) $x^2 + 2x + 1$

Solution: $2x + 2$

(c) $x^2 + e^x$

Solution: $2x + e^x$

48. (3.2) Compute $D\left[-xe^x\right]$ and find a point x where the derivative is zero.

Solution: Use the product rule:
$$D\left[-xe^x\right] = -e^x - xe^x$$
To find the zero derivative:
$$0 = -e^x - xe^x = -(1+x)e^x \implies x = -1$$

49. (3.2) Compute $D\left[xe^{-x}\right]$ without using the chain rule, and find a point x where the derivative is zero.

> **Solution:** Use the quotient rule by noting that $e^{-x} = \frac{1}{e^x}$.
>
> $$D\left[xe^{-x}\right] = D\left[\frac{x}{e^x}\right] = \frac{e^x - xe^x}{e^{2x}} = e^{-x} - xe^{-x}$$
>
> To find the zero derivative:
>
> $$0 = e^{-x} - xe^{-x} = (1-x)e^{-x} \implies x = 1$$

50. (3.2) Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xe^x}{x^2+1}\right]$.

> **Solution:** Use the quotient rule to start the derivative computation
>
> $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xe^x}{x^2+1}\right] = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^x\right](x^2+1) - 2x(xe^x)}{(x^2+1)^2}$$
>
> We must then take the derivative of $\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^x\right]$ in the above numerator using the product rule
>
> $$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^x\right] = e^x + xe^x = (1+x)e^x$$
>
> Substituting this into the quotient rule calculation finishes the derivative:
>
> $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xe^x}{x^2+1}\right] = \frac{(1+x)e^x(x^2+1) - 2x^2e^x}{(x^2+1)^2}$$

51. (3.3) Using the essential trigonometric limits, and the angle sum identity, compute the derivative of $\sin(x)$ using the limit definition. The angle sum identity:

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x).$$

> **Solution:** The limit definition of the derivative of $\sin(x)$ is
>
> $$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}.$$
>
> Simplifying, we get
>
> $$= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h} = \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x).$$
>
> So, the derivative of sine is cosine.

52. (3.3) Compute the derivative of $f(x) = x\sin(x)$

Solution: Using the product rule,

$$f'(x) = \sin(x) + x\cos(x).$$

53. (3.4) Compute the derivative of $f(x) = \left(3x^2 + 2x + 100\right)^{12}$.

Solution: Using the chain rule,

$$f'(x) = 12\left(3x^2 + 2x + 100\right)^{11}(6x + 2) = 24\,(3x + 1)\left(3x^2 + 2x + 100\right)^{11}.$$

54. (3.4) If $y = e^{x^2\sin(x)}$, compute $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution: By the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2\sin(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\sin(x)\right).$$

Using the product rule on the second term, we find that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x^2\sin(x)}\left(x^2\cos(x) + 2x\sin(x)\right) = xe^{x^2\sin(x)}\left(x\cos(x) + 2\sin(x)\right).$$

55. (3.4) If $y = \sin\left(e^{\cos(x)}\right)$, what is $\frac{\mathrm{d}y}{\mathrm{d}x}$?

Solution: By the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos\left(e^{\cos(x)}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\cos(x)}\right].$$

Using the chain rule again,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos\left(e^{\cos(x)}\right)e^{\cos(x)}\left(-\sin(x)\right) = -\sin(x)\cos\left(e^{\cos(x)}\right)e^{\cos(x)}.$$

56. (3.4) Compute $\frac{\mathrm{d}}{\mathrm{d}x}\pi^x$?

Solution:

$$\pi^x = \left(e^{\ln\pi}\right)^x = e^{x\ln\pi}.$$

Hence, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\pi^x = \ln(\pi)e^{x\ln\pi} = \ln(\pi)\pi^x.$$

57. (3.4) Compute $\frac{\mathrm{d}}{\mathrm{d}x}e^{\cos(x) - x^2}$?

Solution: By the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^{\cos(x)-x^2} =& e^{\cos(x)-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(x)-x^2\right]\\ =& e^{\cos(x)-x^2}\left(-\sin(x)-2x\right)\\ =& -e^{\cos(x)-x^2}\left(\sin(x)+2x\right).\end{aligned}$$

58. (3.5) Consider the curve defined by the equation $\frac{x^2}{4}+\frac{y^2}{9}=8$.

(a) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$?

Solution: We determine the derivative through implicit differentiation

$$\frac{2x}{4}+\frac{2y}{9}\frac{\mathrm{d}y}{\mathrm{d}x}=0.$$

Rearranging, we find that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2x}{2y}\frac{9}{4}=-\frac{9x}{4y}.$$

(b) Demonstrate that the point $(4,-6)$ lies on this curve.

Solution: We substitute $x=4$, $y=-6$ into the equation, and find that

$$\frac{x^2}{4}+\frac{y^2}{9}=\frac{4^2}{4}+\frac{(-6)^2}{9}=4+4=8.$$

Since $(4,-6)$ satisfies the curve equation, it must lie on the curve.

(c) Find the equation of the tangent line at the point $(4,-6)$?

Solution: The gradient of the tangent line is equal to the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point. That is, the gradient is

$$m=-\frac{9}{4}\frac{4}{-6}=\frac{3}{2}.$$

Using the point-slope formula for a line, the tangent has equation

$$y+6=\frac{3}{2}\left(x-4\right).$$

Rearranging this equation, we find that the tangent has equation

$$y=\frac{3}{2}x-12$$

59. (3.5) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x^2\cos(y)=e^y+\ln(x)$.

Solution: We differentiate implicitly, using the product rule on the left hand side, so

$$2x\cos(y) - x^2\sin(y)\frac{\mathrm{d}y}{\mathrm{d}x} = e^y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}.$$

This means that

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(-e^y - x^2\sin(y)\right) = \frac{1}{x} - 2x\cos(y),$$

and therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^{-1} - 2x\cos(y)}{e^y + x^2\sin(y)}.$$

60. (3.5) Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation, when $x^2y + e^{x+y} = -230$.

Solution: We differentiate implicitly, finding that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2y\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{x+y}\right] = 0.$$

We use the product rule on the first term, and the chain rule on the second term, and find that

$$2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} + e^{x+y}\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0.$$

This means that

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(x^2 + e^{x+y}\right) = -2xy - e^{x+y},$$

and therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy + e^{x+y}}{x^2 + e^{x+y}}.$$

61. (3.6) Let $g(z) = \cos(z)$, over the domain $[-\pi, 0]$.

(a) Briefly explain why g on this domain is invertible.

Solution: This is because cos is one-to-one over the domain $[-\pi, 0]$. You can see this through sketching the graph of $\cos(x)$ over this domain: any horizontal line intersects at most once.

(b) Let $h(x) = g^{-1}(x)$ be the inverse function. What is the range of h?

Solution: The range of h is the domain of g. This is $[-\pi, 0]$.

(c) Demonstrate using implicit differentiation that $h'(x) = -\frac{1}{\sin(h(x))}$.

Solution: Write $y = h(x)$. Taking cos of both sides, we find that $\cos(y) = \cos(h(x))$. Since h is an inverse of g, and g is equal to cos over its domain, it must be that $\cos(h(x)) = x$. We thus find that $\cos(y) = x$. By implicit differentiation,

$$-\sin(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$

and therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin(y)}$. Since $y = h(x)$, we find that

$$h'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin\left(h(x)\right)}.$$

(d) Using a trigonometric identity, show that $h'(x) = -\frac{1}{\sqrt{1-x^2}}$.

Solution: Using Pythagoras' Theorem, $\sin\left(h(x)\right) = \sqrt{1 - \cos\left(h(x)\right)^2}$. Now $\cos\left(h(x)\right) = x$, since h is the inverse of g, and g is equal to cos over its domain. This means that

$$h'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}}.$$

62. (3.6) Compute $\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1}\left(x^3\right)$?

Solution: By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1}\left(x^3\right) = \frac{1}{1+(x^3)^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right).$$

We thus find that

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1}\left(x^3\right) = \frac{3x^2}{1+x^6}.$$

63. (3.6) Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin^{-1}(x)}{1+x^2}\right]$?

Solution: Using the quotient rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin^{-1}(x)}{1+x^2}\right] &= \frac{(1+x^2)\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) - 2x\sin^{-1}(x)}{(1+x^2)^2} \\ &= \left(1+x^2\right)^{-1}\frac{1}{\sqrt{1-x^2}} - \frac{2x\sin^{-1}(x)}{(1+x^2)^2} \\ &= \frac{1}{(1+x^2)\sqrt{1-x^2}} - \frac{2x\sin^{-1}(x)}{(1+x^2)^2}.\end{aligned}$$

64. (3.7) Derive the natural logarithm derivative identity $(\ln(x))' = \frac{1}{x}$, by performing implicit differentiation of $\ln^{-1}(y) = e^y = x$.

Solution: The natural log is defined as the inverse of the exponential: $\ln(x) = y \iff e^y = x$. We implicitly differentiate y with the latter expression:

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{y(x)} = \frac{\mathrm{d}}{\mathrm{d}x}x$$

yielding

$$e^y y' = 1.$$

Solving for y' and using the fact that $e^y = x$:

$$y' = \frac{1}{e^y} = \frac{1}{x}.$$

65. (3.7) Compute the derivatives of the following functions involving logarithms

(a) $f(x) = \ln(x^2 + 1)$

Solution: f is a composition of ln and $x^2 + 1$, so the chain rule is required:

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x^2 + 1) = \frac{1}{x^2 + 1} \frac{\mathrm{d}}{\mathrm{d}x} \left[x^2 + 1\right] = \frac{2x}{x^2 + 1}.$$

(b) $f(x) = x \log_{10}(x)$

Solution: f is a product of $\log_{10}$ and x, so the product rule is required:

$$\frac{\mathrm{d}}{\mathrm{d}x} [x \log_{10}(x)] = \log_{10}(x) + x \frac{1}{\ln(10)x} = \log_{10}(x) + \frac{1}{\ln(10)}.$$

(c) $f(x) = \ln(\ln(x))$, $x > 1$

Solution: f is a composition of two log functions so the chain rule must be used:

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(\ln(x)) = \frac{1}{\ln(x)} \frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x \ln(x)}.$$

(d) $f(x) = \tan^{-1}(\ln(x))$

Solution: f is a composition of inverse tan and log so the chain rule must be used:

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan^{-1}(\ln(x)) = \frac{1}{1 + (\ln(x))^2} \frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x\,(1 + (\ln(x))^2)}.$$

66. (3.9) Find the linear approximation function to the following functions at the specified points $x = a$.

(a) $f(x) = \tan^{-1}(x)$, $x = 0$.

Solution: $f(0) = 0$, $f'(x) = \frac{1}{1+x^2}$, $f'(0) = 1$.

$$f(x) \approx 0 + 1(x - 0) = x.$$

(b) $f(x) = \frac{1}{1+x^2}$, $x = -1$.

Solution: $f(-1) = 1/2$, $f'(x) = \frac{-2x}{(1+x^2)^2}$, $f'(-1) = \frac{2}{4} = \frac{1}{2}$.

$$f(x) \approx \frac{1}{2} + \frac{1}{2}(x+1) = \frac{x}{2} + 1.$$

67. (3.9) Consider the equation for an ellipse: $1 = x^2 + \frac{1}{4}y^2$.

(a) Verify that the parametric functions $x(\theta) = \cos(\theta)$ and $y(\theta) = 2\sin(\theta)$ satisfy the ellipse equation for all angles θ.

Solution: "Verifying" means substituting $x(\theta)$ and $y(\theta)$ into the ellipse equation and simplifying until you can state facts from trigonometry that are true for any θ value you choose: Note,

$$x(\theta)^2 + \frac{1}{4}y(\theta)^2 = \cos^2(\theta) + \frac{1}{4}(2\sin(\theta))^2 = \cos^2(\theta) + \sin^2(\theta).$$

The well-known pythagorean identity states that $\cos^2(\theta) + \sin^2(\theta) = 1$ for any θ-value, so the ellipse equation is satisfied by the given parametric equations.

(b) Find a linear approximation in the x-y plane to the elliptic curve at the x-y point defined by $\theta = \pi/4$.

Solution: At $\theta = \pi/4$, $x = \sqrt{2}/2$ and $y = \sqrt{2}$. One can solve the equation for y to find a function of x, then take the derivative to find the slope of the linear approximation, or one can use implicit differentiation to find the slope. Either approach will work, but here we will use implicit differentiation:

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}1 = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + \tfrac{1}{4}y^2\right] = 2x + \frac{1}{2}yy'.$$

Solving for y', we get

$$y' = \frac{-2x}{\frac{1}{2}y} = -\frac{4x}{y}.$$

Evaluating the expression at $x = \sqrt{2}/2$ and $y = \sqrt{2}$, results in

$$y' = -\frac{4\sqrt{2}/2}{\sqrt{2}} = -2.$$

So, the linear approximation is

$$y(x) = \sqrt{2} - 2(x - \sqrt{2}/2).$$

68. (3.9) Suppose you know the following facts about the function $f(x)$:

(i) f is differentiable everywhere,

(ii) $f(2) = 2$,

(iii) $f'(2) = -3$.

Give your best estimate of the value of $f(x)$ at $x = 2.2$.

Solution: The best estimate given the known facts is given by the linear approximation:

$$f(x) \approx f(2) + f'(2)(x-2) = 2 - 3(x-2),$$

so

$$f(2.2) \approx 2 - 3(.2) = 1.4$$

69. (3.9) Use the linear approximation of $f(x) = \sqrt{x}$ to estimate the value of $\sqrt{99}$.

Solution: We know that $f(100) = \sqrt{100} = 10$ is a known point nearby $x = 99$. Also,

$$f'(x) = \frac{1}{2\sqrt{x}},$$

so,

$$f(99) \approx 10 + \frac{1}{20}(99 - 100) = 10 - \frac{1}{20} = 10 - 0.05 = 9.95 \approx \sqrt{99}.$$

Note, the ten-decimal-point value is $\sqrt{99} = 9.949874371066199$. The linear approximation is accurate to the first two digits.

70. (3.9) Use the small-angle approximations of sine and cosine to estimate the value of $\tan(0.1)$.

Solution:

$$\tan(0.1) = \frac{\sin(0.1)}{\cos(0.1)} \approx \frac{0.1}{1} = 0.1.$$

71. (4.1) An electrical circuit contains two variable-resistance resistors in series, with resistances R_1 and R_2 given in Ω. It is known that the total resistance R may be expressed as

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

The first resistance R_1 is equal to $\frac{4}{3}\,\Omega$, and is decreasing at a rate of $4\,\frac{\Omega}{\mathrm{h}}$. The second resistance R_2 is equal to $4\,\Omega$ and is increasing at a rate of $8\,\frac{\Omega}{\mathrm{h}}$.

(a) What is the total resistance R?

Solution: We first evaluate R.

$$\frac{1}{R} = \frac{3}{4}\frac{1}{\Omega} + \frac{1}{4}\frac{1}{\Omega} = 1\,\frac{1}{\Omega}.$$

This means that $R = 1\,\Omega$.

(b) What is the rate of change of R in $\frac{\Omega}{\mathrm{h}}$?

Solution: By the Chain Rule,

$$-\frac{1}{R^2}\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{1}{R_1^2}\frac{\mathrm{d}R_1}{\mathrm{d}t} - \frac{1}{R_2^2}\frac{\mathrm{d}R_2}{\mathrm{d}t}.$$

Substituting in the values for R, R_1 and R_2,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{9}{16}\frac{\mathrm{d}R_1}{\mathrm{d}t} + \frac{1}{16}\frac{\mathrm{d}R_2}{\mathrm{d}t} = -\frac{9\cdot 4}{16} + \frac{8}{16} = -\frac{7}{4}.$$

This means that the total resistance is decreasing at a rate of $\frac{7}{4}\,\frac{\Omega}{\mathrm{h}}$.

72. (4.2) Let $f(x) = x^{\frac{1}{3}}(x-1)^2$. Identify the critical numbers of f.

Solution: The critical numbers are where $f' = 0$, or the derivative doesn't exist. Differentiating,

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}(x-1)^2 + 2(x-1)x^{\frac{1}{3}} = x^{\frac{1}{3}}(x-1)\left(\frac{x-1}{3x} + 2\right).$$

Since $|f'(x)| \to \infty$ as $x \to 0$, the derivative does not exist at $x = 0$, so $x = 0$ is a critical number.
The derivative is zero when either $x - 1 = 0$, in which case $x = 1$, or

$$\frac{x-1}{3x} + 2 = 0,$$

in which case $x = \frac{1}{7}$.
In summary, the critical numbers $0, 1$ and $\frac{1}{7}$.

73. (4.2) Let $f(x) = \cos(x) + 1 - x$. Find the absolute maximum and absolute minimum values of f over the interval $[-\pi, 2\pi]$.

Solution: We first identify the local maxima and minima through differentiating. We find that

$$f'(x) = -\sin(x) - 1.$$

Over the interval, $f'(x) = 0$ when $x = -\frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.
Evaluating the function at these points, we find that $f\left(-\frac{\pi}{2}\right) = 1 + \frac{\pi}{2}$ and $f\left(\frac{3\pi}{2}\right) = 1 - \frac{3\pi}{2}$.
At the ends of the interval, $f(-\pi) = \pi$, and $f(2\pi) = 2 - 2\pi$.
Of these four values, the biggest is π, so the absolute maximum is π.
The smallest is $2 - 2\pi$, so the absolute minimum is $2 - 2\pi$.

74. (4.3) Let $f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 2x + 10$.

(a) Find the extreme points of f. Identify which of these are local maxima and which are local minima, and explain your answer.

Solution: Differentiating,

$$f'(x) = 2x^2 - 3x - 2 = (x-2)(2x+1).$$

Since f is differentiable everywhere, its extreme values are only where its derivative is zero. These points are 2 and $-\frac{1}{2}$.
There are two ways you can determine whether these extreme values are maxima or minima. The first method is to check the second derivative. The second derivative is

$$f''(x) = 4x - 3.$$

Since $f'' > 0$ at $x = 2$, this is a local minimum. Since $f'' < 0$ at $x = -\frac{1}{2}$, this is a local maximum.
The other method is to look at the signs of f' either side of the extreme value. It can be checked that $f' > 0$ for $x > 2$, and $f'(0) = -2 < 0$. This means that $x = 2$ is a local minimum. Also $f'(-1) = 3$, and since $f'(0) < 0$, this means that $x = -\frac{1}{2}$ is a local maximum.

(b) Find the intervals over which f is increasing, and the intervals where f is decreasing.

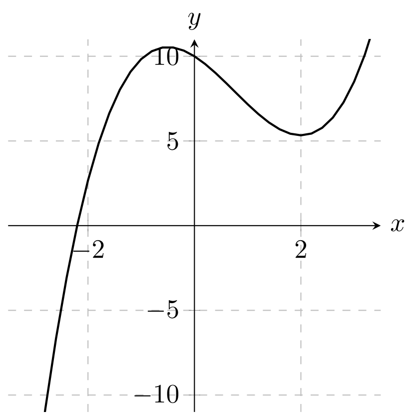
Solution: f is increasing over the interval $(2, \infty)$, because $x = 2$ is a local minimum. f is decreasing over the interval $(-1/2, 2)$, because this interval has a local maximum on the left and a local minimum on the right. f is increasing over the interval $(-\infty, -1/2)$, since this interval has a local maximum on the right.

(c) Find any inflection points, and intervals where f is concave upwards, and intervals where f is concave downwards.

Solution: Inflection points can only occur when $f'' = 0$. The only point where $f'' = 0$ is $x = \frac{3}{4}$. This is an inflection point because $f'' < 0$ for $x < \frac{3}{4}$, and $f'' > 0$ for $x > \frac{3}{4}$.
The function is concave upwards on the interval $(\frac{3}{4}, \infty)$, and it is concave downwards over the interval $(-\infty, \frac{3}{4})$.

(d) Sketch the curve.

Solution:



75. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x \to 3} \frac{x-3}{\ln(4-x)}$$

Solution: The indeterminate form is "$\frac{0}{0}$" and it is

$$\lim_{x \to 3} \frac{x-3}{\ln(4-x)} \overset{l'H}{=} \lim_{x \to 3} \frac{1}{-\frac{1}{4-x}} = \lim_{x \to 3} -(4-x) = -1$$

76. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x\to\infty} \frac{e^x}{\ln(x)}$$

Solution: The indeterminate form is "$\frac{\infty}{\infty}$" and it is

$$\lim_{x\to\infty} \frac{e^x}{\ln(x)} \stackrel{l'H}{=} \lim_{x\to\infty} \frac{e^x}{\frac{1}{x}} = \lim_{x\to\infty} xe^x = \infty$$

77. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x\to\frac{\pi}{2}^-} (x - \frac{\pi}{2}) \tan(x)$$

Solution: The indeterminate form is "$0 \cdot \infty$" and it is

$$\lim_{x\to\frac{\pi}{2}^-} (x - \frac{\pi}{2}) \tan(x) = \lim_{x\to\frac{\pi}{2}^-} \frac{(x - \frac{\pi}{2}) \sin(x)}{\cos(x)} \stackrel{l'H}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\sin(x) + (x - \frac{\pi}{2}) \cos(x)}{-\sin(x)} = -1$$

78. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x\to-2} \left(1 + \frac{2}{x}\right)^{x+2}$$

Solution: The indeterminate form is "0^0" and applying ln to the limit gives

$$\begin{aligned}\lim_{x\to-2} \ln\left(\left(1 + \frac{2}{x}\right)^{x+2}\right) &= \lim_{x\to-2} (x+2) \ln\left(1 + \frac{2}{x}\right) = \lim_{x\to-2} \frac{\ln\left(1 + \frac{2}{x}\right)}{\frac{1}{x+2}} \\ &\stackrel{l'H}{=} \lim_{x\to-2} \frac{\frac{1}{1+\frac{2}{x}}\left(-\frac{2}{x^2}\right)}{-\frac{1}{(x+2)^2}} = \lim_{x\to-2} \frac{(x+2)^2 2}{\left(1 + \frac{2}{x}\right) x^2} \\ &= \lim_{x\to-2} \frac{2(x+2)^2}{x(x+2)} = \lim_{x\to-2} \frac{2(x+2)}{x} = 0\end{aligned}$$

Reversing the ln gives

$$\lim_{x\to-2} \left(1 + \frac{2}{x}\right)^{x+2} = e^0 = 1$$

79. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x\to0^+} (1 + \sin(2x))^{\frac{1}{x}}$$

Solution: The indeterminate form is "1^∞" and applying ln to the limit gives

$$\lim_{x\to 0^+} \ln\left((1+\sin(2x))^{\frac{1}{x}}\right) = \lim_{x\to 0^+} \frac{1}{x}\ln(1+\sin(2x)) = \lim_{x\to 0^+} \frac{\ln(1+\sin(2x))}{x}$$

$$\overset{l'H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+\sin(2x)}2\cos(2x)}{1} = \lim_{x\to 0^+} \frac{2\cos(2x)}{1+\sin(2x)} = 2$$

Reversing the ln gives

$$\lim_{x\to 0^+} (1+\sin(2x))^{\frac{1}{x}} = e^2$$

80. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x\to\infty} \left(\frac{x^2+4}{x}\right)^{\frac{1}{\ln(x)}}$$

Solution: The indeterminate form is "∞^0" and applying ln to the limit gives

$$\lim_{x\to\infty} \ln\left(\left(\frac{x^2+4}{x}\right)^{\frac{1}{\ln(x)}}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x^2+4}{x}\right)}{\ln(x)} \overset{l'H}{=} \lim_{x\to\infty} \frac{\frac{x}{x^2+4}\,\frac{2x^2-(x^2+4)}{x^2}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{x^2-4}{x^2+4} = \lim_{x\to\infty} \frac{1-\frac{4}{x^2}}{1+\frac{4}{x^2}} = 1$$

Reversing the ln gives

$$\lim_{x\to\infty} \left(\frac{x^2+4}{x}\right)^{\frac{1}{\ln(x)}} = e^1 = e$$

81. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan(x)}\right)$$

Solution: The indeterminate form is "$\infty - \infty$" and it is

$$\lim_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan(x)}\right) = \lim_{x\to 0^+} \frac{\sin(x) - x^2\cos(x)}{x^2\sin(x)}$$

$$\overset{l'H}{=} \lim_{x\to 0^+} \frac{\cos(x) - 2x\cos(x) + x^2\sin(x)}{2x\sin(x) + x^2\cos(x)} = \infty$$

82. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x\to\infty} (\ln(x+1) - \ln(\ln(x)))$$

Solution: The indeterminate form is "$\infty - \infty$" and it is

$$\lim_{x\to\infty} (\ln(x+1) - \ln(\ln(x))) = \lim_{x\to\infty} \ln\left(\frac{x+1}{\ln(x)}\right) = \ln\left(\lim_{x\to\infty} \frac{x+1}{\ln(x)}\right)$$
$$\stackrel{l'H}{=} \ln\left(\lim_{x\to\infty} \frac{1}{\frac{1}{x}}\right) = \ln\left(\lim_{x\to\infty} x\right) = \infty$$

83. (4.5) Identify the indeterminate form and compute the limit.

$$\lim_{x\to 0} \frac{\sin(x)^2}{x\ln(1-x)}$$

Solution: The indeterminate form is "$\frac{0}{0}$" and it is

$$\lim_{x\to 0} \frac{\sin(x)^2}{x\ln(1-x)} \stackrel{l'H}{=} \lim_{x\to 0} \frac{2\sin(x)\cos(x)}{-\frac{x}{1-x} + \ln(1-x)} = \lim_{x\to 0} \frac{2\sin(x)\cos(x)(1-x)}{-x + (1-x)\ln(1-x)}$$

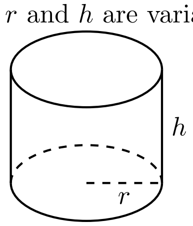
The indeterminate form is "$\frac{0}{0}$" and it is

$$\lim_{x\to 0} \frac{2\sin(x)\cos(x)(1-x)}{-x+(1-x)\ln(1-x)} \stackrel{l'H}{=} \lim_{x\to 0} \frac{2\left(\cos(x)^2(1-x) - \sin(x)^2(1-x) - \sin(x)\cos(x)\right)}{-1-\ln(1-x)-1}$$
$$= -1$$

84. (4.6) A canned soup firm wants to redesign their 1 l size cans, so that they minimize the materials used. They want to give it the shape of a cylinder. Find the dimensions of the new can.

(a) Assign symbols to the relative quantities. Which quantity is to be optimized? Which quantities are variables?

Solution: Let r be the radius and h the height of the can. Let S be the surface and $V = 1\,\text{l}$ the volume. S is to be minimized and r and h are variables.



(b) Find the objective function, that is a function for the to be optimized quantity.

Solution: $S = 2\pi r^2 + 2\pi rh$

(c) Find a constraint equation, that is an equation relating all the variable quantities, such that the objective function only depends on one variable quantity.

Solution: The volume is constant and

$$V = \pi r^2 h \qquad h = \frac{V}{\pi r^2} \qquad S = 2\pi r^2 + \frac{2V}{r} \quad 0 < r$$

(d) Optimize the objective function.

Solution:

$$S'(r) = 4\pi r - \frac{2V}{r^2} = \frac{4\pi}{r^2}\left(r^3 - \frac{V}{2\pi}\right)$$

So $r = \sqrt[3]{\frac{V}{2\pi}}$ is the critical point. It is the minimum, since $S(r) \to \infty$ for $r \to \infty$ or $r \to 0$.

(e) Relate your solution to the original question.

Solution: The surface of the can is minimized for $r = \sqrt[3]{\frac{V}{2\pi}} \approx 0.276\,\text{dm}$ and $h = \sqrt[3]{\frac{4V}{\pi}} \approx 2.205\,\text{dm}$.

85. (4.6) Find two numbers a, b such that $a^2 + b^2 = 25$ and the sum is maximal.

Solution: The sum $S = a + b$ is to be maximized. The constraint function $a^2 + b^2 = 25$ gives

$$b = \sqrt{25 - a^2} \qquad S(a) = a + \sqrt{25 - a^2} \qquad -5 \le a \le 5$$

Now maximizing $S(a)$:

$$S'(a) = 1 + \frac{-2a}{2\sqrt{25 - a^2}} = \frac{\sqrt{25 - a^2} - a}{\sqrt{25 - a^2}} \qquad a^2 = 25 - a^2 \qquad a = \frac{5}{\sqrt{2}}$$

Thus $a = \frac{5}{\sqrt{2}}$ is the critical point. It is a maximum, since

$$S(-5) = -5 \qquad S(5) = 5 \qquad S(\frac{5}{\sqrt{2}}) = \frac{10}{\sqrt{2}} > 5$$

Thus $a = \frac{5}{\sqrt{2}} = b$ maximize the sum.

86. (4.7) Find an algorithm to approximate $\sqrt[4]{12}$.

(a) Find a polynomial function with integer coefficients, for which $\sqrt[4]{12}$ is a root.

Solution: $\sqrt[4]{12}$ is a root of $f(x) = x^4 - 12$.

(b) Give the iteration formula and an initial approximation.

Solution:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 12}{4x_n^3} = \frac{3x_n^4 + 12}{4x_n^3}$$

Initial guesses could be $x_1 = 1$ or $x_1 = 2$. Not possible are negative numbers, since they converge to $-\sqrt[4]{12}$ and 0, since the tangent line at $x = 0$ is horizontal.

87. (4.7) Explain, why Newton's method does not work for finding a root of $f(x) = 2x^3 - 3x^2 - 12x + 18$, if the initial approximation is $x_1 = 2$.

Solution:

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$$

So $f'(x_1) = 0$. Thus the linear approximation at $x = 2$ does not intersect the x-axis and Newton's method does not work.

88. (4.7) Use Newton's method to approximate the positive root of $f(x) = x^2 - 7$.

(a) Give the iteration formula.

Solution:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 7}{2x_n} = \frac{x_n^2 + 7}{2x_n}$$

(b) Perform two iterations using the initial approximation $x_1 = 1$.

Solution:

$$x_1 = 1 \qquad x_2 = \frac{1+7}{2} = 4 \qquad x_3 = \frac{16+7}{8} = \frac{23}{8} = 2.875$$

89. (4.8) Verify, that $(x^2 - 2x + 2)e^x$ is an antiderivative of x^2e^x.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 2x + 2)e^x = (2x - 2)e^x + (x^2 - 2x + 2)e^x = x^2e^x$$

Thus $(x^2 - 2x + 2)e^x$ is an antiderivative of x^2e^x.

90. (4.8) Find an antiderivative of the following functions.

(a) $f(x) = 3x^4 + 7$

Solution:

$$F(x) = \frac{3}{5}x^5 + 7x$$

(b) $f(x) = \sin(x)$

Solution:

$$F(x) = -\cos(x)$$

(c) $f(x) = \frac{1}{x} + e^x$

Solution:

$$F(x) = \ln|x| + e^x$$

91. (4.8) Find the most general antiderivative of the following functions.

(a) $f(x) = 3x^2 - e^x$

Solution:
$$F(x) = x^3 - e^x + C$$

(b) $f(x) = \cos(x) - \frac{1}{x^2}$

Solution:
$$F(x) = \sin(x) + \frac{1}{x}$$

92. (4.8) Consider the function $f(x) = 2\sin(x)\cos(x)$.

(a) Verify, that $F_1(x) = -\cos(x)^2$ is an antiderivative of $f(x)$.

Solution:
$$F_1'(x) = -2\cos(x)(-\sin(x)) = 2\cos(x)\sin(x) = f(x)$$

(b) Verify, that $F_2(x) = \sin(x)^2$ is an antiderivative of $f(x)$.

Solution:
$$F_2'(x) = 2\sin(x)\cos(x) = f(x)$$

(c) How is it possible, that $F_1(x)$ and $F_2(x)$ are both antiderivatives of $f(x)$.

Solution: It is
$$F_1(x) = -\cos(x)^2 = \sin(x)^2 - 1 = F_2(x) - 1$$
and thus they differ by a constant. The antiderivative is only unique up to a constant.

93. (5.1) Using summation "sigma" notation, write down the sum S of sequential integers 3 through 12 in the following ways:

(a) Using an index that starts at $n = 3$ and ends at $n = 12$. Specify exactly your indexed set A.

Solution: $S = \sum_{n=3}^{12} n$, $A = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} = \{n\}_{n=3}^{12}$.

(b) Using an index that starts at $n = 0$ and ends at $n = 9$.Specify exactly your indexed set A.

Solution: $S = \sum_{n=0}^{9}(3 + n)$, $A = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} = \{(n + 3)\}_{n=0}^{9}$.

94. (5.1) Using summation "sigma" notation, write down the sum S of sequential multiples of 5 starting at 25 through 50. Write out the index set $A = \{a_n\}$ and the explicit sum "$\sum$" indicating starting and ending indices n. There are many correct ways to index and sum.

Solution:

$$A = \{25, 30, 35, 40, 45, 50\} = \{5n + 20\}_{n=1}^{6}$$

$$S = \sum_{n=1}^{6} (5n + 20)$$

95. (5.1) With three rectangles, approximate the area under $f(x) = \sin(x)^2$ between $x = 0$ and $x = \frac{3\pi}{2}$ using right endpoints.

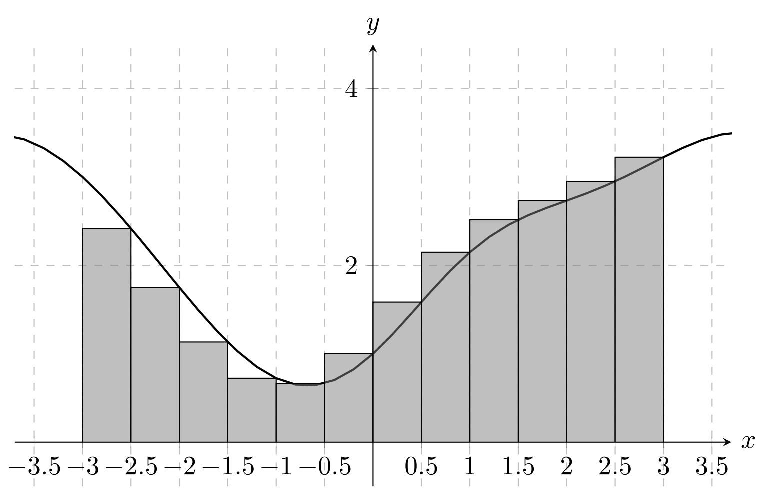
Solution: With three rectangles

$$\Delta x = \frac{\frac{3\pi}{2} - 0}{3} = \frac{\pi}{2} \qquad x_i = \frac{\pi}{2} + i\frac{\pi}{2}$$

Therefore

$$A \approx R_3 = \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) = \frac{\pi}{2}\left(\sin(\pi/2)^2 + \sin(\pi)^2 + \sin(3\pi/2)^2\right)$$
$$= \frac{\pi}{2}\left(1^2 + 0 + (-1)^2\right) = \pi$$

96. (5.1) Consider the given function $f(x)$ and the rectangles approximating its area on $[-3, 3]$. Give an expression for the area of the rectangles in the figure below. Use $\sum$ notation to express it.

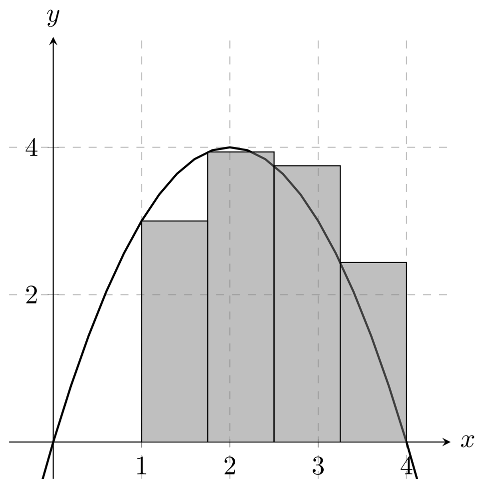


Solution:

$$Area = \frac{1}{2}f(-2.5) + \frac{1}{2}f(-2) + \cdots + \frac{1}{2}f(3) = \sum_{i=1}^{12} \frac{1}{2} f(-3 + \frac{i}{2})$$

97. (5.1) Consider the given function $f(x) = -(x-2)^2 + 4$ on $[1, 4]$. Find an estimate of the area under the graph of f on $[1, 4]$ by using four rectangles. Use the left endpoints as sample points.

(a) Draw the rectangles.



(b) Find the area of the rectangles.

Solution:

$$\Delta x = \frac{4-1}{4} = \frac{3}{4} = 0.75$$

$$\begin{aligned} Area =& \frac{3}{4}f(1) + \frac{3}{4}f(1.75) + \frac{3}{4}f(2.5) + \frac{3}{4}f(3.25) \\ =& \frac{3}{4}\left((-1+4) + \left(-\frac{1}{16}+4\right) + \left(-\frac{1}{4}+4\right) + \left(-\frac{25}{16}+4\right)\right) \\ =& \frac{3}{4}\left(15 - \frac{15}{8}\right) = \frac{3\cdot 15\cdot 7}{4\cdot 8} = \frac{315}{32} = 9.84375 \end{aligned}$$

98. (5.1) The table below gives velocity of a particle at 4 time points. Using this data, approximate the distance travelled by the particle using the left endpoints.

time (s)	0	1	2	3
velocity ($\frac{\text{ft}}{\text{s}}$)	1	1	3	4

Solution: Using left endpoints:

$$d = (t_1 - t_0)v_0 + (t_2 - t_1)v_1 + (t_3 - t_2)v_2 = 1\cdot 1 + 1\cdot 1 + 1\cdot 3 = 5\,\text{ft}$$

99. (5.2) Consider $\displaystyle\int_1^3 \frac{2-x}{4}\mathrm{d}x$

(a) Express this integral as the limit of a sum. You can pick your favourite sample points, e.g. right endpoints.

Solution: If we use the right endpoints and $\Delta x = \frac{2}{n}$, then $x_i = 1 + i\frac{2}{n}$ and

$$\int_1^3 \frac{2-x}{4}\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2-\left(1+i\frac{2}{n}\right)}{4}\frac{2}{n}$$
$$= \lim_{n\to\infty}\frac{1}{2n}\sum_{i=1}^{n}\left(1-i\frac{2}{n}\right)$$

(b) Evaluate the integral using a geometric argument. (Hint: What does the function look like?)

Solution: The function is a line that passes through zero at 2. Since $x = 2$ is the midpoint between domain of integration, $\int_1^3 \frac{2-x}{4}\mathrm{d}x = 0$.

100. (5.2) For each of the following Riemann Sums, identify the definite integral it converges to (but don't evaluate the integral). In addition, identify whether the left-endpoint, right-endpoint or midpoint Riemann Sum has been used.

(a) $\displaystyle\lim_{n\to\infty}\frac{12}{n}\sum_{j=0}^{n-1}e^{2x_j}\left(x_j - x_j^2\right)$, where $x_j = -4 + \frac{12j}{n}$.

Solution: This converges to

$$\int_{-4}^{8} e^{2x}(x-x^2)\mathrm{d}x.$$

The integral starts at -4 because $x_0 = -4$, and the intervals are spaced $\frac{12}{n}$ apart (since $x_{j+1} - x_j = \frac{12}{n}$), which means that the end point of the integration must be $-4+12 = 8$. The left endpoint rule has been used because the first evaluation is at $x_0 = -4$ (i.e. the left of the first interval), and the last evaluation is at $x_{n-1} = 8 - \frac{12}{n}$, which is the left of the last interval.

(b) $\displaystyle\lim_{n\to\infty}\frac{9}{n}\sum_{j=1}^{n}\left(3x_j - x_j^4\right)\cos(x_j)$, where $x_j = -3 + \frac{9j}{n}$.

Solution: This converges to

$$\int_{-3}^{6}(3x - x^4)\cos(x)\mathrm{d}x.$$

The integral starts at -3 because $x_1 = -3 + \frac{9}{n} \to -3$ as $n \to \infty$, and the intervals are spaced $\frac{9}{n}$ apart (since $x_{j+1} - x_j = \frac{9}{n}$), which means that the end point of the integration must be $-3+9 = 6$. The right endpoint rule has been used because the first evaluation is at $x_1 = -3+\frac{9}{n}$ (i.e. the right of the first interval), and the last evaluation is at $x_n = 6$, which is the right of the last interval.

(c) $\displaystyle\lim_{n\to\infty}\frac{10}{n}\sum_{j=0}^{n-1}\left(3x_j - x_j^4\right)\log(x_j)$, where $x_j = 3 + \frac{10j+5}{n}$.

Solution: This converges to

$$\int_{3}^{13}(3x - x^4)\log(x)\mathrm{d}x.$$

The integral starts at 3 because $x_0 \to 3$ as $n \to \infty$, and the intervals are spaced $\frac{10}{n}$ apart (since $x_{j+1} - x_j = \frac{10}{n}$), which means that the end point of the integration must be $3 + 10 = 13$. The midpoint rule has been used because for example the first evaluation is halfway between 3 and $3 + \frac{10}{n}$, i.e $3 + \frac{5}{n}$.

101. (5.2) Write down the Riemann sum for the integral $\int_2^4 x^2 \mathrm{d}x$ using a right-endpoint-rule with equally-spaced partitions Δx_n.

Solution: Let N be the number of partitions. The interval $[2, 4]$ is broken up by the index set $\{x_n = 2 + \frac{2}{N} n\}_{n=1}^N$. Each interval has width $\Delta x_n = \frac{2}{N}$. The approximate sum is

$$\sum_{n=1}^{N} f(x_n)\Delta x_n = \sum_{n=1}^{N} (2 + \tfrac{2}{N} n)^2 \tfrac{2}{N}.$$

The Riemann sum is then

$$\lim_{N\to\infty} \sum_{n=1}^{N} (2 + \tfrac{2}{N} n)^2 \tfrac{2}{N} = \int_2^4 x^2 \mathrm{d}x.$$

102. (5.3) Compute the following definite integrals:

(a) $\displaystyle\int_2^4 (x^2 - x)\,\mathrm{d}x$

Solution: $\displaystyle\int_2^4 (x^2 - x)\,\mathrm{d}x = \frac{1}{3}x^3 - \frac{1}{2}x^2\Big|_2^4 = \frac{64}{3} - 8 - \left(\frac{8}{3} - 2\right) = \frac{56}{3} - 6 = \frac{38}{3}$

(b) $\displaystyle\int_{-1}^1 (e^x - 3)\,\mathrm{d}x$

Solution: $\displaystyle\int_{-1}^1 (e^x - 3)\,\mathrm{d}x = e - \frac{1}{e} - 6$

(c) $\displaystyle\int_1^e \frac{1}{x}\,\mathrm{d}x$

Solution: $\displaystyle\int_1^e \frac{1}{x}\mathrm{d}x = \ln(e) - \ln(1) = 1$

103. (5.3) Consider the function

$$f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}.$$

(a) Verify the function is continuous everywhere, particularly at $x = 1$.

Solution: Polynomials are continuous everywhere, and note x and x^2 share the same value at $x = 1$, so f is continuous everywhere.

(b) Find an antiderivative $F^-(x)$ of $f(x)$ at points $x \leq 1$.

Solution: $F^-(x) = \frac{1}{2}x^2$

(c) Find an antiderivative $F^+(x)$ of $f(x)$ at points $x > 1$.

Solution: $F^+(x) = \frac{1}{3}x^3$

(d) Compute $\displaystyle\int_0^1 f(x)\mathrm{d}x$.

Solution: $\int_0^1 f(x)\mathrm{d}x = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$

(e) Compute $\displaystyle\int_1^2 f(x)\mathrm{d}x$.

Solution: $\int_1^2 f(x)\mathrm{d}x = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}(8-1) = \frac{7}{3}$

(f) Explain why $\displaystyle\int_0^2 f(x)\mathrm{d}x$ exists, but is *not* equal to $F^+(2) - F^-(0)$.

Solution: $F^+(2) - F^-(0) = (\frac{1}{3}x^3)\big|_2 - (\frac{1}{2}x^2)\big|_0 = \frac{8}{3}$; however, the antiderivative of f over the interval $[0,2]$ is *not* equal to either of the antiderivatives on the subintervals $[0,1]$, or $[1,2]$. The area under the curve is

$$\int_0^2 f(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x + \int_1^2 f(x)\mathrm{d}x = \frac{1}{2} + \frac{7}{3} = \frac{17}{6}.$$

104. (5.3) Consider the function

$$f(x) = \begin{cases} 0 & x < 1 \\ 2 & 1 \leq x \leq 3 \\ 1/2 & 3 < x \leq 7 \\ 5 & 7 \leq x \end{cases}$$

Compute the following integrals.

(a) $\displaystyle\int_{-1}^3 f(x)\mathrm{d}x$

Solution: $\displaystyle\int_{-1}^3 f(x)\mathrm{d}x = 2 \cdot (3-1) = 4$

(b) $\displaystyle\int_0^9 f(x)\mathrm{d}x$

Solution: $\displaystyle\int_0^9 f(x)\mathrm{d}x = (1-0)\cdot 0 + (3-1)\cdot 2 + (7-3)\cdot \frac{1}{2} + (9-7)\cdot 5 = 16$

(c) $\int_{1/2}^{3/2} f(x)\mathrm{d}x$

> **Solution:** $\int_{1/2}^{3/2} f(x)\mathrm{d}x = \left(\frac{3}{2} - 1\right) \cdot 2 = 1$

105. Consider the function $f(x)$ defined on $[-1, 3]$

$$f(x) = \begin{cases} -1, & x \leq 1 \\ x - 2, & 1 < x \leq 3 \end{cases}.$$

Use the FTC part 1 to find an antiderivative $F(x)$ of $f(x)$ defined on all of $[-1, 3]$.

> **Solution:** The function $f(x)$ is piecewise defined, so its antiderivative will be piecewise defined as well. Given $-1 \leq x \leq 1$, then
>
> $$F(x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{x} (-1)dt = [-t]_{-1}^{x} = -x - 1.$$
>
> for $x > 1$, then
>
> $$F(x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{1} (-1)dt + \int_{1}^{x} (t-2)dt = -2 + [\tfrac{1}{2}t^2 - 2t]_1^x = -2 + \tfrac{1}{2}x^2 - 2x - (\tfrac{1}{2} - 2) = \tfrac{1}{2}x^2 - 2x - \tfrac{1}{2}.$$
>
> An antiderivative defined for all of $[-1, 3]$ is then
>
> $$F(x) = \begin{cases} -x - 1, & x \leq 1 \\ \frac{1}{2}x^2 - 2x - \frac{1}{2}, & 1 < x \leq 3 \end{cases}.$$

106. Compute the derivative of

$$g(x) = \int_{3}^{x} (2t^2 - 3t + 1)dt.$$

> **Solution:** By the FTC pt. 1: $g'(x) = (2x^2 - 3x + 1)$.

107. (5.5) Evaluate the following integrals using substitution.

(a) $\int (12 - 4x)^{51} \mathrm{d}x$

> **Solution:**
>
> $$u = 12 - 4x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -4 \qquad -\frac{1}{4}\mathrm{d}u = \mathrm{d}x$$
>
> $$\int (12 - 4x)^{51} \mathrm{d}x = \int u^{51} \left(-\frac{1}{4}\right) \mathrm{d}u = -\frac{1}{4} \cdot \frac{1}{52} u^{52} + C = -\frac{1}{208} (12 - 4x)^{52} + C$$

(b) $\int x^2 e^{2x^3} \mathrm{d}x$

Solution:

$$u = 2x^3 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 6x^2 \qquad \frac{1}{6}\mathrm{d}u = x^2 \mathrm{d}x$$

$$\int x^2 e^{2x^3} \mathrm{d}x = \int e^u \frac{1}{6} \mathrm{d}u = \frac{1}{6} e^u + C = \frac{1}{6} e^{2x^3} + C$$

(c) $\int \sin(x)\cos(x)^4 \mathrm{d}x$

Solution:

$$u = \cos(x) \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin(x) \qquad -\mathrm{d}u = \sin(x)\mathrm{d}x$$

$$\int \sin(x)\cos(x)^4 \mathrm{d}x = \int u^4(-1)\mathrm{d}u = -\frac{1}{5}u^5 + C = -\frac{1}{5}\cos(x)^5 + C$$

108. (5.5) Decide for the following integrals whether the given substituion is successful. If yes, then solve the integral using this substituion.

(a) $\int \left(e^{\cos(x)} + \sin(x)\right) \mathrm{d}x$ with $u = \cos(x)$

Solution: The substitution is not successful, since the $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin(x)$ is not a factor of the integrand.

(b) $\int \frac{1 + \ln(x)}{x} \mathrm{d}x$ with $u = \ln(x)$

Solution: The substitution is successfully

$$u = \ln(x) \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \qquad \mathrm{d}u = \frac{1}{x}\mathrm{d}x$$

$$\int \frac{1 + \ln(x)}{x} \mathrm{d}x = \int (1 + u)\,\mathrm{d}u = u + \frac{1}{2}u^2 + C = \ln(x) + \frac{1}{2}\ln(x)^2 + c$$

(c) $\int \frac{\sin(x)}{\cos(x)} \mathrm{d}x$ with $u = \sin(x)$

Solution: The substitution is not successful, since $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos(x)$ is not a factor of the integrand (it is in the denominator, but it would need to be in the numerator).

109. (5.5) Given

$$\int_0^4 f(x)\mathrm{d}x = 15$$

Find the following integrals.

(a) $\int_0^2 x f(x^2) \mathrm{d}x$

Solution: Choose $u = x^2$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and $\frac{1}{2}\mathrm{d}u = x\mathrm{d}x$. So it is

$$\int_0^2 xf(x^2)\mathrm{d}x = \int_{u(0)}^{u(2)} f(u)\frac{1}{2}\mathrm{d}u = \frac{1}{2}\int_0^4 f(u)\mathrm{d}u = \frac{15}{2}$$

(b) $\int_{-4}^{0} f(-x)\mathrm{d}x$

Solution: Choose $u = -x$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = -1$ and $-\mathrm{d}u = \mathrm{d}x$. So it is

$$\int_{-4}^{0} f(-x)\mathrm{d}x = \int_4^0 f(u)(-1)\mathrm{d}u = \int_0^4 f(u)\mathrm{d}u = 15$$

(c) $\int_{-2}^{2} f(x+2)\mathrm{d}x$

Solution: Choose $u = x + 2$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\mathrm{d}u = \mathrm{d}x$. So it is

$$\int_{-2}^{2} f(x+2)\mathrm{d}x = \int_0^4 f(u)\mathrm{d}u = 15$$

110. (5.5) Evaluate the following integrals using substitution.

(a) $\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\mathrm{d}x$

Solution:

$$u = \sqrt{x} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \qquad 2\mathrm{d}u = \frac{1}{\sqrt{x}}\mathrm{d}x$$

$$\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\mathrm{d}x = \int_0^{\pi} \sin(u)2\mathrm{d}u = -2\cos(u)|_0^{\pi} = 4$$

(b) $\int_{-1}^{1} \frac{e^x}{1+e^x}\mathrm{d}x$

Solution:

$$u = 1 + e^x \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = e^x \qquad \mathrm{d}u = e^x\mathrm{d}x$$

$$\int_{-1}^{1} \frac{e^x}{1+e^x}\mathrm{d}x = \int_{1+\frac{1}{e}}^{1+e} \frac{1}{u}\mathrm{d}u = \ln|u||_{1+\frac{1}{e}}^{1+e} = \ln|1+e| - \ln\left|1+\frac{1}{e}\right|$$

111. (5.5) Decide for the following integrals whether the given substitution is helpful for solving the integral.

(a) $\int xe^x\mathrm{d}x$ with $u = x$

Solution: This substitution is not helpful, since $\mathrm{d}u = \mathrm{d}x$ and the integral does not change.

(b) $\displaystyle\int \frac{e^x}{x}\mathrm{d}x$ with $x = \ln(u)$

Solution: The substitution is not helpful, since $\mathrm{d}x = \frac{1}{u}\mathrm{d}u$ and so

$$\int \frac{e^x}{x}\mathrm{d}x = \int \frac{u}{\ln(u)}\frac{1}{u}\mathrm{d}u = \int \frac{1}{\ln(u)}\mathrm{d}u$$

This integral is not easier than the original one.

112. (5.6) Evaluate the following integrals using integration by parts.

(a) $\displaystyle\int xe^x\mathrm{d}x$

Solution:

$$f(x) = x \quad f'(x) = 1 \qquad g'(x) = e^x \quad g(x) = e^x$$

$$\int xe^x\mathrm{d}x = xe^x - \int 1e^x\mathrm{d}x = xe^x - e^x + C$$

(b) $\displaystyle\int x\ln(x)\mathrm{d}x$

Solution:

$$f(x) = \ln(x) \quad f'(x) = \frac{1}{x} \qquad g'(x) = x \qquad g(x) = \frac{1}{2}x^2$$

$$\int x\ln(x)\mathrm{d}x = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x^2\frac{1}{x}\mathrm{d}x = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$$

113. (5.6) Evaluate the following integrals using integration by parts multiple times.

(a) $\displaystyle\int x^2\sin(x)\mathrm{d}x$

Solution:

$$f(x) = x^2 \quad f'(x) = 2x \qquad g'(x) = \sin(x) \quad g(x) = -\cos(x)$$

$$\int x^2\sin(x)\mathrm{d}x = -x^2\cos(x) - \int 2x(-\cos(x))\mathrm{d}x = -x^2\cos(x) + 2\int x\cos(x)\mathrm{d}x$$

$$f(x) = x \quad f'(x) = 1 \qquad g'(x) = \cos(x) \quad g(x) = \sin(x)$$

$$\int x^2\sin(x)\mathrm{d}x = -x^2\cos(x) + 2x\sin(x) - 2\int 1\sin(x)\mathrm{d}x = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

(b) $\displaystyle\int 1\cdot\ln(x)^2\mathrm{d}x$

Solution:

$$f(x) = \ln(x)^2 \quad f'(x) = 2\ln(x)\frac{1}{x} \qquad g'(x) = 1 \qquad g(x) = x$$

$$\int 1 \cdot \ln(x)^2 \mathrm{d}x = x\ln(x)^2 - \int x 2\ln(x)\frac{1}{x}\mathrm{d}x = x\ln(x)^2 - 2\int \ln(x)\mathrm{d}x$$

$$f(x) = \ln(x) \quad f'(x) = \frac{1}{x} \qquad g'(x) = 1 \quad g(x) = x$$

$$\int 1 \cdot \ln(x)^2 \mathrm{d}x = x\ln(x)^2 - 2x\ln(x) + 2\int x\frac{1}{x}\mathrm{d}x = x\ln(x)^2 - 2x\ln(x) + 2x + C$$

114. (5.6) Decide for the following integrals whether the given setup for integration by parts works and helps to solve the integral.

(a) $\int x\ln(x)\mathrm{d}x$ with $f(x) = x$ and $g'(x) = \ln(x)$

Solution: Integration by parts does not work, since finding $g(x)$ is too hard.

(b) $\int x\cos(x)\mathrm{d}x$ with $f(x) = \cos(x)$ and $g'(x) = x$

Solution: Integration by parts works, but it gives

$$\int x\cos(x)\mathrm{d}x = \frac{1}{2}x^2\cos(x) - \int \frac{1}{2}x^2(-\sin(x))\mathrm{d}x$$

and the resulting integral is more complicated than the integral at the beginning.

115. (5.7) Use algebraic transformations to evaluate the following integrals.

(a) $\int \frac{1}{x^2+16}\mathrm{d}x$

Solution:

$$\int \frac{1}{x^2+16}\mathrm{d}x = \int \frac{1}{16\left(\frac{x^2}{16}+1\right)}\mathrm{d}x = \frac{1}{16}\int \frac{1}{\left(\frac{x}{4}\right)^2+1}\mathrm{d}x$$

Use substitution

$$u = \frac{x}{4} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{4} \qquad 4\mathrm{d}u = \mathrm{d}x$$

$$\frac{1}{16}\int \frac{1}{\left(\frac{x}{4}\right)^2+1}\mathrm{d}x = \frac{1}{16}\int \frac{1}{u^2+1}4\mathrm{d}u = \frac{1}{4}\arctan(u) + C = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C$$

(b) $\int \frac{1}{x^2-4x+8}\mathrm{d}x$

Solution: $x^2 - 4x + 8$ does not have any real solutions, since the quadratic formula gives

$$x = 2 \pm \sqrt{4-8} = 2 \pm \sqrt{-4}$$

So complete the square

$$x^2 - 4x + 8 = x^2 - 4x + 4 + 4 = (x-2)^2 + 4$$

This gives

$$\int \frac{1}{x^2-4x+8}\mathrm{d}x = \int \frac{1}{(x-2)^2+4}\mathrm{d}x$$

Using substitution

$$u = x - 2 \qquad \mathrm{d}u = \mathrm{d}x$$

$$\int \frac{1}{(x-2)^2+4}\mathrm{d}x = \int \frac{1}{u^2+4}\mathrm{d}u = \int \frac{1}{4\left(\frac{u^2}{4}+1\right)}\mathrm{d}u = \frac{1}{4}\int \frac{1}{\left(\frac{u}{2}\right)^2+1}\mathrm{d}u$$

Using substitution

$$v = \frac{u}{2} \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1}{2} \qquad 2\mathrm{d}v = \mathrm{d}u$$

$$\frac{1}{4}\int \frac{1}{\left(\frac{u}{2}\right)^2+1}\mathrm{d}u = \frac{1}{4}\int \frac{1}{v^2+1}2\mathrm{d}v = \frac{1}{2}\arctan(v) + C = \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$$
$$= \frac{1}{2}\arctan\left(\frac{x-2}{2}\right) + C$$

(c) $\displaystyle\int \frac{4x+3}{2x-1}\mathrm{d}x$

Solution:

$$\int \frac{4x+3}{2x-1}\mathrm{d}x = \int \frac{4x-2+5}{2x-1}\mathrm{d}x = \int \left(\frac{4x-2}{2x-1} + \frac{5}{2x-1}\right)\mathrm{d}x = \int \left(2 + \frac{5}{2x-1}\right)\mathrm{d}x$$

Use substitution on the second summand

$$u = 2x - 1 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad \frac{1}{2}\mathrm{d}u = \mathrm{d}x$$

$$\int \left(2 + \frac{5}{2x-1}\right)\mathrm{d}x = \int 2\mathrm{d}x + \int \frac{5}{u}\frac{1}{2}\mathrm{d}u = 2x + \frac{5}{2}\ln|u| + C = 2x + \frac{5}{2}\ln|2x-1| + C$$

116. (5.7) Evaluate the following integrals using partial fractions.

(a) $\displaystyle\int \frac{x+10}{(x+3)(x-4)}\mathrm{d}x$

Solution:

$$\frac{x+10}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4} \qquad x + 10 = A(x-4) + B(x+3) = Ax - 4A + Bx + 3B$$

$$1 = A + B \qquad 10 = -4A + 3B \qquad 14 = 7B \qquad B = 2 \qquad A = -1$$

$$\int \frac{x+10}{(x+3)(x-4)}\mathrm{d}x = \int \left(-\frac{1}{x+3} + \frac{2}{x-4}\right)\mathrm{d}x = -\ln|x+3| + 2\ln|x-4| + C$$

(b) $\displaystyle\int \frac{5x^2+21x+1}{(x+2)^2(x-1)}\mathrm{d}x$

Solution:

$$\frac{5x^2+21x+1}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}$$

$$5x^2+21x+1 = A(x+2)(x-1) + B(x-1) + C(x+2)^2 = Ax^2 + Ax - 2A + Bx - B + Cx^2 + 4Cx + 4C$$

$$1 = -2A - B + 4C \qquad 21 = A + B + 4C \qquad 5 = A + C$$

$$A = 5-C \qquad 1 = -10+2C-B+4C \qquad 11 = -B+6C \qquad 21 = 5-C+B+4C \qquad 16 = B+3C$$

$$27 = 9C \qquad C = 3 \qquad A = 2 \qquad B = 16 - 3C = 7$$

$$\int \frac{5x^2 + 21x + 1}{(x+2)^2(x-1)}\mathrm{d}x = \int \left(\frac{2}{x+2} + \frac{7}{(x+2)^2} + \frac{3}{x-1}\right)\mathrm{d}x = 2\ln|x+2| - \frac{7}{x+2} + 3\ln|x-1| + C$$

(c) $\int \frac{6x^2 - 12x - 4}{(x^2+4)(x-1)}\mathrm{d}x$

Solution:

$$\frac{6x^2 - 12x - 4}{(x^2+4)(x-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1}$$

$$6x^2 - 12x - 4 = (Ax+B)(x-1) + C(x^2+4) = Ax^2 - Ax + Bx - B + Cx^2 + 4C$$

$$-4 = -B + 4C \qquad -12 = -A + B \qquad 6 = A + C$$

$$B = 4 + 4C \qquad -12 = -A + 4 + 4C \qquad -16 = -A + 4C$$

$$-10 = 5C \qquad C = -2 \qquad B = -4 \qquad A = 6 - C = 8$$

$$\int \frac{6x^2 - 12x - 4}{(x^2+4)(x-1)}\mathrm{d}x = \int \left(\frac{8x}{x^2+4} - \frac{4}{x^2+4} - \frac{2}{x-1}\right)\mathrm{d}x$$

For the first summand use substitution

$$u = x^2 + 4 \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \qquad \mathrm{d}u = 2x\mathrm{d}x$$

$$\int \frac{8x}{x^2+4}\mathrm{d}x = \int \frac{4}{u}\mathrm{d}u = 4\ln|u| + C = 4\ln|x^2+4| + C$$

For the second summand

$$\frac{4}{x^2+4} = \frac{4}{4\left(\frac{x^2}{4}+1\right)} = \frac{1}{\left(\frac{x}{2}\right)^2 + 1}$$

Use substitution

$$v = \frac{x}{2} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{2} \qquad 2\mathrm{d}v = \mathrm{d}x$$

$$\int \frac{4}{x^2+4}\mathrm{d}x = \int \frac{1}{v^2+1}2\mathrm{d}v = 2\arctan(v) + C = 2\arctan\left(\frac{x}{2}\right) + C$$

So it is

$$\int \left(\frac{8x}{x^2+4} - \frac{4}{x^2+4} - \frac{2}{x-1}\right)\mathrm{d}x = 4\ln|x^2+4| - 2\arctan\left(\frac{x}{2}\right) - 2\ln|x-1| + C$$

117. (5.7) Evaluate the following integrals using trigonometric identities.

(a) $\int \cos(x)^3\mathrm{d}x$

Solution:

$$\int \cos(x)^3\mathrm{d}x = \int \cos(x)\left(1 - \sin(x)^2\right)\mathrm{d}x$$

Use substitution

$$u = \sin(x) \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos(x) \qquad \mathrm{d}u = \cos(x)\mathrm{d}x$$

$$\int \cos(x)\left(1-\sin(x)^2\right)\mathrm{d}x = \int \left(1-u^2\right)\mathrm{d}u = u - \frac{1}{3}u^3 + C = \sin(x) - \frac{1}{3}\sin(x)^3 + C$$

(b) $\int \cos(x)^4 \mathrm{d}x$

Solution:

$$\int \cos(x)^4\mathrm{d}x = \int \left(\frac{1}{2}\left(1+\cos(2x)\right)\right)^2 \mathrm{d}x = \frac{1}{4}\int \left(1+2\cos(2x)+\cos(2x)^2\right)\mathrm{d}x$$

$$= \frac{1}{4}\int \left(1+2\cos(2x)+\frac{1}{2}\left(1+\cos(4x)\right)\right)\mathrm{d}x$$

$$= \frac{1}{4}\left(x+\sin(2x)+\frac{1}{2}x+\frac{1}{8}\sin(4x)\right)+C$$

(c) $\int \sin(x)^4\cos(x)^3\mathrm{d}x$

Solution:

$$\int \sin(x)^4\cos(x)^3\mathrm{d}x = \int \sin(x)^4\left(1-\sin(x)^2\right)\cos(x)\mathrm{d}x$$

Use substitution

$$u = \sin(x) \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos(x) \qquad \mathrm{d}u = \cos(x)\mathrm{d}x$$

$$\int \sin(x)^4\left(1-\sin(x)^2\right)\cos(x)\mathrm{d}x = \int u^4\left(1-u^2\right)\mathrm{d}u = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \frac{1}{5}\sin(x)^5 - \frac{1}{7}\sin(x)^7 + C$$

118. (5.7) Evaluate the following integrals using trigonometric substitution.

(a) $\int \sqrt{16-x^2}\mathrm{d}x$

Solution: Use substitution

$$x = 4\sin(u) \qquad \frac{\mathrm{d}x}{\mathrm{d}u} = 4\cos(u) \qquad \mathrm{d}x = 4\cos(u)\mathrm{d}u$$

$$\int \sqrt{16-x^2}\mathrm{d}x = \int \sqrt{16-16\sin(u)^2}4\cos(u)\mathrm{d}u = 16\int \cos(u)^2\mathrm{d}u$$

Using a trigonometric identity

$$16\int \cos(u)^2\mathrm{d}u = 16\int \frac{1}{2}\left(1+\cos(2u)\right)\mathrm{d}u = 8\left(u+\frac{1}{2}\sin(2u)\right)+C$$

Now to resubstitute

$$u = \arcsin\left(\frac{x}{4}\right) \qquad \sin(2u) = 2\sin(u)\cos(u) = 2\frac{x}{4}\sqrt{1-\left(\frac{x}{4}\right)^2}$$

$$8\left(u+\frac{1}{2}\sin(2u)\right)+C=\frac{1}{8}\left(\arcsin\left(\frac{x}{4}\right)+\frac{x}{4}\sqrt{1-\left(\frac{x}{4}\right)^2}\right)+C$$

(b) $\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

Solution: Use substitution

$$x=\sin(u) \qquad \frac{\mathrm{d}x}{\mathrm{d}u}=\cos(u) \qquad \mathrm{d}x=\cos(u)\mathrm{d}u$$

$$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x=\int \frac{1}{\cos(u)}\cos(u)\mathrm{d}u=\int 1\mathrm{d}u=u+C=\arcsin(x)+C$$

119. Compute the following improper integral, if it exists, by computing as a limit of proper integral

$$\int_0^\infty \frac{1}{1+x^2}dx.$$

Solution: First set the integral up as a limit of proper integral:

$$\int_0^\infty \frac{1}{1+x^2}dx=\lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx.$$

Compute the proper integral and then take the limit:

$$\lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx=\lim_{t\to\infty}\left[\tan^{-1}(x)\right]_0^t$$
$$=\lim_{t\to\infty}\left(\tan^{-1}(t)-\tan^{-1}(0)\right)=\lim_{t\to\infty}\tan^{-1}(t)=\tfrac{\pi}{2}.$$

120. Compute the following improper integral, if it exists, by computing as a limit of proper integrals

$$\int_0^\infty \frac{1}{1+x}dx.$$

Solution: First set the integral up as a limit of proper integrals:

$$\int_0^\infty \frac{1}{1+x}dx=\lim_{t\to\infty}\int_0^t \frac{1}{1+x}dx.$$

Compute the proper integral and then take the limit:

$$\lim_{t\to\infty}\int_0^t \frac{1}{1+x}dx=\lim_{t\to\infty}\left[\ln(x+1)\right]_0^t$$
$$=\lim_{t\to\infty}\left(\ln(t+1)-\ln(1)\right)=\lim_{t\to\infty}\ln(t+1)=\infty.$$

The limit diverges to infinity, so the value of the improper integral does not exist.

121. Compute the following improper integral, if it exists, by computing as a limit of proper integrals

$$\int_1^9 \frac{1}{\sqrt[3]{x-1}} dx.$$

Solution: The integral is improper because the integrand is not defined at $x = 1$, therefore, we set it up as a limit of proper integrals where it is defined:

$$\int_1^9 \frac{1}{\sqrt[3]{x-1}} dx = \lim_{t \to 1^+} \int_t^9 \frac{1}{\sqrt[3]{x-1}} dx.$$

Compute the proper integral and then take the limit:

$$\begin{aligned} \lim_{t \to 1^+} \int_t^9 (x-1)^{-\frac{1}{3}} dx &= \lim_{t \to 1^+} \left[\tfrac{3}{2}(x-1)^{\frac{2}{3}}\right]_t^9 \\ &= \tfrac{3}{2}(8)^{\frac{2}{3}} - \lim_{t \to 1^+} \tfrac{3}{2}(t-1)^{\frac{2}{3}} \\ &= \tfrac{12}{2} = 6. \end{aligned}$$

122. Compute the following improper integral, if it exists, by computing as a limit of proper integrals

$$\int_{-\infty}^{\infty} 2x dx.$$

Solution: Set it up as a limit of proper integrals:

$$= \lim_{w \to -\infty} \int_w^0 2x dx + \lim_{t \to \infty} \int_0^t 2x dx,$$

where

$$= \lim_{w \to -\infty} [x^2]_w^0 + \lim_{t \to \infty} [x^2]_0^t,$$

evaluating

$$= \lim_{w \to -\infty} -w^2 + \lim_{t \to \infty} t^2 = -\infty + \infty = DNE.$$

Neither of the above two integral limits exist (are finite) so the full improper integral does not exist either. Note, $\infty - \infty$ is not defined, nor is this situation analogous to a L'Hospital's-type limit because each of the limits must be computed (and exist) independently of the other, which is why it's important to use distinct limit variables t and w to emphasize that the two limits cannot be yoked together in a way that makes them appear to cancel each other.

123. Compute the following improper integral, if it exists, by computing as a limit of proper integrals

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx.$$

Solution: Set it up as a limit of proper integrals:

$$= \lim_{w\to-\infty} \int_w^0 \frac{x}{1+x^2}dx + \lim_{t\to\infty} \int_0^t \frac{x}{1+x^2}dx$$

The indefinite integral has a solution via u-subtitution

$$u = 1 + x^2, \quad \frac{du}{2x} = dx,$$
$$\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{1}{u}du$$
$$= \frac{1}{2}\ln(1+x^2) + C.$$

Using the above result the proper integrals become

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx = \lim_{w\to-\infty} [\tfrac{1}{2}\ln(1+x^2)]_w^0 + \lim_{t\to\infty} [\tfrac{1}{2}\ln(1+x^2)]_0^t$$
$$= \lim_{t\to\infty} (\tfrac{1}{2}\ln(1+t^2)) - \lim_{w\to-\infty} (\tfrac{1}{2}\ln(1+w^2)) = \infty - \infty = DNE!$$

Neither of the two integral limits exist (are finite) so the full improper integral does not exist either. Note, $\infty - \infty$ is not defined, nor is this situation analogous to a L'Hospital's-type limit because each of the limits must be computed (and exist) independently of the other, which is why it's important to use distinct limit variables t and w to emphasize that the two limits cannot be yoked together in a way that makes them appear to cancel each other.

124. Compute the following improper integral, if it exists, by computing as a limit of proper integrals

$$\int_{-\infty}^{\infty} xe^{-x^2}dx.$$

Solution: Set it up as a limit of proper integrals:

$$= \lim_{w\to-\infty} \int_w^0 xe^{-x^2}dx + \lim_{t\to\infty} \int_0^t xe^{-x^2}dx$$

The integral $\int xe^{-x^2}dx$ can be solved with u-substitution:

$$u = -x^2, \quad -\frac{du}{2x} = dx$$
$$\implies \int xe^{-x^2}dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^{-x^2} + C.$$

So,

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \lim_{w\to-\infty} [-\frac{1}{2}e^{-x^2}]_w^0 + \lim_{t\to\infty} [-\frac{1}{2}e^{-x^2}]_0^t$$

and

$$= -\frac{1}{2} + \frac{1}{2} + \lim_{w\to-\infty} \frac{1}{2}e^{-w^2} - \lim_{t\to\infty} \frac{1}{2}e^{-t^2} = 0 + 0 - 0 = 0.$$

Each limit exists independently so the improper integral exists.

125. Compute the following improper integral, if it exists, by computing as a limit of proper integrals

$$\int_{-\infty}^{\infty} e^{-|x|}\sin(x)dx.$$

Solution: Set it up as a limit of proper integrals:

$$= \lim_{w\to-\infty}\int_{w}^{0} e^{-|x|}\sin(x)dx + \lim_{t\to\infty}\int_{0}^{t} e^{-|x|}\sin(x)dx$$

On the $x<0$ domain, the integral can be re-expressed by substituting $u=-x$, $-du=dx$:

$$\lim_{w\to-\infty}\int_{w}^{0} e^{-|x|}\sin(x)dx = -\lim_{w\to-\infty}\int_{-w}^{0} e^{-u}\sin(u)du = -\lim_{w\to\infty}\int_{0}^{w} e^{-u}\sin(u)du.$$

So, the improper integral can be expressed as

$$\int_{-\infty}^{\infty} e^{-|x|}\sin(x)dx = \lim_{t\to\infty}\int_{0}^{t} e^{-x}\sin(x)dx - \lim_{w\to\infty}\int_{0}^{w} e^{-u}\sin(u)du.$$

The above two integral terms appear to be equal but opposite sign and it's tempting to conclude that the improper integral is zero, but remember that the two integral limits must exist in order for the two terms to cancel. So, we must verify that the limit

$$\lim_{t\to\infty}\int_{0}^{t} e^{-x}\sin(x)dx$$

exists first, before we conclude the improper integral is zero. Computing the antiderivative involves two iterations of integration by parts

$$\begin{aligned}\int e^{-x}\sin(x)dx &= -e^{-x}\sin(x) + \int e^{-x}\cos(x)dx\\ &= -e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\sin(x)dx\\ \implies 2\int e^{-x}\sin(x)dx &= -e^{-x}\sin(x) - e^{-x}\cos(x)\\ \implies \int e^{-x}\sin(x)dx &= -\frac{1}{2}e^{-x}(\sin(x)+\cos(x)).\end{aligned}$$

So,

$$\begin{aligned}\lim_{t\to\infty}\int_{0}^{t} e^{-x}\sin(x)dx &= \frac{1}{2} + \lim_{t\to\infty} -\frac{1}{2}e^{-t}(\sin(t)+\cos(t))\\ &= \frac{1}{2} - 0 = \frac{1}{2}.\end{aligned}$$

The limit exists, so

$$\int_{-\infty}^{\infty} e^{-|x|}\sin(x)dx = \frac{1}{2} - \frac{1}{2} = 0.$$