Mapping the Earth

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4. Outline.

- Maps of the Earth
- Curvilinear Coordinates of the Sphere
  - Spherical Coordinates
- Metric of the Sphere
  - Angles and Lengths using Metric
  - Change of Coordinates
- Equation of Isometric Mapping Sphere to Plane
  - Necessary Condition: Curvature Vanishes.
  - Curvature of Sphere Does Not Vanish.
  - Zero Curvature is Sufficient for Existence of Map
- Equiareal Maps
  - Lambert, Sanson, Boone & Mollweide Projections.
- Equiangular (Conformal) Maps
  - Mercator & Stereographic Projections.
- Milnor's Theorem About Maps with Least Distortion
  - Azimuthal Equidistant Projection.
- Chebychev's Theorem About Least Distortion Conformal Maps.

5. Mapping Regions on the Surface of the Earth.

The Differential Geometry of Curves and Surfaces is no longer offered as a regular undergraduate course. I hope to give a sampling from this intriguing and important subject.

Let's assume that the surface of the earth is a unit sphere

$$\mathbb{S}^2 = \{X \in \mathbb{R}^3 : |X|^2 = 1\}$$

We imagine trying to draw a map of a region $D \subset \mathbb{S}^2$. The map is a smooth function

$$F : D \to E^2$$

to the Euclidean plane that preserves as much information about the geometry of D as is possible.

We shall see that it is impossible to make a map that preserves all distances, b an isometric mapping that preserves lengths of all curves. However, we may construct a mapping that preserves angles, be a conformal mapping or preserves areas, be an equiareal mapping.

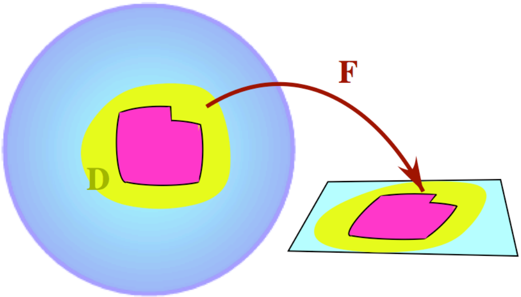


Figure 1: Mapping $X : D \to \mathbf{E}^2$ of a Region

We seek a mapping of the region with as little distortion as possible.

The sphere can locally be given curvilinear coordinates, also called a parameterization. Let $\Omega \subset \mathbf{R}^2$ be open. Let

$$X : \Omega \to \mathbb{S}^2$$

be a smooth function. Then we want $D = X(\Omega)$ to be a piece of the sphere. At each point $P \in D$ we can identify tangent vectors to the surface. If $P = X(a)$ some $a \in \Omega$, then

$$X_i(a) = \frac{\partial X}{\partial u^i}(a)$$

for $i = 1, 2$ are vectors in $\mathbf{R}^3$ tangent to the coordinate curves. To avoid singularities at P, we shall assume that all $X_1(P)$and $X_2(P)$ are linearly independent vectors. Then the tangent plane to the surface at P is

$$T_P M = \operatorname{span}\{X_1(P), X_2(P)\}.$$

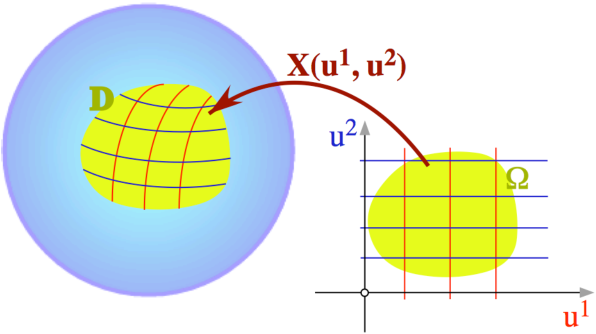


Figure 2: Coordinate chart $X : \Omega \to D$ on sphere.

The sphere may be calculated in lots of different coordinate charts. The geometric quantities such as length, angle and curvature should not depend on the choice of chart.

Spherical coordinates give one example of local coordinates. $(u^1, u^2) \in \Omega = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi)$ give the latitude and longitude in radians

$$X(u^1, u^2) = (\cos u^1 \ \cos u^2, \ \cos u^1 \ \sin u^2, \ \sin u^1).$$

where $D = \{(x, y, z) \in \mathbb{S}^2 : \ y \neq 0 \text{ if } x \leq 0\}$.
The tangent vectors are thus

$$\begin{aligned} X_1(u^1, u^2) &= \left(-\sin u^1 \ \cos u^2, \ -\sin u^1 \ \sin u^2, \ \cos u^1\right), \\ X_2(u^1, u^2) &= \left(-\cos u^1 \ \sin u^2, \ \cos u^1 \ \cos u^2, \ 0\right) \end{aligned} \tag{1}$$

which are linearly independent for every $(u^1, u^2) \in \Omega$.

A mapping of a region $D \subset \mathbb{S}^2$ is a function $f : D \to \mathbb{E}^2$ to the Euclidean plane.

If D is a coordinate chart given by $X : \Omega \to D$, then each point is the image $X(u^1, u^2) \in D$ of some point $(u^1, u^2) \in \Omega \subset \mathbf{R}^2$. Thus the mapping is given by

$$f(X(u)) = f(X(u^1, u^2)) = (v^1(u^1, v^1), v^2(u^1, v^1)) = v(u)$$

where $v : \Omega \to \mathbb{E}^2$ is a function. Usually we assume v is a diffeomeorphism.

Figure 3: Orthographic Projection.

Orthographic projection is the view of the earth from infinity $f : (x, y, z) \to (x, z)$ from the $y > 0$ hemisphere. In terms of spherical coordinates $X(u^1, u^2) =$

$$(\cos u^1 \ \cos u^2, \ \cos u^1 \ \sin u^2, \ \sin u^1),$$

On $0 < u^2 < \pi$ the map is

$$v(u) = (\cos u^1 \ \cos u^2, \ \sin u^1).$$

The Euclidean structure of $\mathbf{R}^3$, the usual dot product, gives a way to measure lengths and angles of vectors. If $V = (v_1, v_2, v_3)$ then its length

$$|V| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{V \bullet V}$$

If $W = (w_1, w_2, w_3)$ then the angle $\alpha = \angle(V, W)$ is given by

$$\cos\alpha = \frac{V \bullet W}{|V|\,|W|}.$$

If $\gamma : [a, b] \to M \subset \mathbf{R}^3$ is a continuously differentiable curve, its length is

$$\mathrm{L}(\gamma) = \int_a^b |\dot{\gamma}(t)|\, dt.$$

If the curve is confined to a coordinate patch $\gamma([a,b]) \subset X(\Omega) \subset \mathbb{S}^2$, then we may factor through the coordinate chart. There are continuously differentiable $u(t) = (u_1(t), u_2(t)) \in \Omega$ so that

$$\gamma(t) = X(u_1(t), u_2(t)) \qquad \text{for all } t \in [a,b].$$

Then the tangent vector may be written

$$\dot\gamma(t) = X_1(u_1(t), u_2(t))\, \dot u_1(t) + X_2(u_1(t), u_2(t))\, \dot u_2(t)$$

so its length is

$$|\dot\gamma|^2 = X_1 \bullet X_1\, \dot u_1^2 + 2X_1 \bullet X_2\, \dot u_1 \dot u_2 + X_2 \bullet X_2\, \dot u_2^2$$

For $i, j = 1, 2$ the Riemannian Metric is is given by the matrix function

$$g_{ij}(u) = X_i(u) \bullet X_j(u)$$

Evidently, $g_{ij}(u)$ is a smoothly varying, symmetric and positive definite.

Thus $|\dot\gamma(t)|^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}(u(t))\,\dot u_i(t)\,\dot u_j(t).$

The length of the curve on the surface is determined by its velocity in the coordinate patch $\dot u(t)$ and the metric $g_{ij}(u)$.

A vector field on the surface is also determined by functions in U using the basis. Thus if V and W are tangent vector fields, they may be written

$$V(u) = v^1(u)X_1(u) + v^2(u)X_2(u), \qquad W(u) = w^1(u)X_1(u) + w^2(u)X_2(u)$$

The $\mathbf{R}^3$ dot product can also be expressed by the metric. Thus

$$V \bullet W = \langle V, W\rangle = \sum_{i,j=1}^{2} g_{ij}\, v^i\, w^j.$$

where $\langle\cdot,\cdot\rangle$ is an inner product on T_pM that varies smoothly on M. This Riemannian metric is also called the First Fundamental Form.

If V and W are nonvanishing vector fields on M then their angle $\alpha = \angle(V, W)$ satisfies
$$\cos\alpha = \frac{\langle V, W\rangle}{|V|\,|V|}$$
which depends on coordinates of the vector fields and the metric.
If $\omega \subset \Omega$ is a piecewise smooth subdomain in the patch, the area if $X(\omega) \subset \mathbb{S}^2$ is also determined by the metric
$$\mathrm{A}(X(\omega)) = \int_\omega |X_1 \times X_2|\, du_1\, du_2 = \int_\omega \sqrt{\det(g_{ij}(u))}\, du_1\, du_2$$
since if $\beta = \angle(X_1, X_2)$ then
$$\begin{aligned} |X_1 \times X_2|^2 &= \sin^2\beta\, |X_1|^2\, |X_2|^2 = (1 - \cos^2\beta)\, |X_1|^2\, |X_2|^2 \\ &= |X_1|^2\, |X_2|^2 - (X_1 \bullet X_2)^2 = g_{11}g_{22} - g_{12}^2. \end{aligned}$$

Tangent vectors are

$$X_1(u^1,u^2) = \left(-\sin u^1\ \cos u^2,\ -\sin u^1\ \sin u^2,\ \cos u^1\right),$$
$$X_2(u^1,u^2) = \left(-\cos u^1\ \sin u^2,\ \cos u^1\ \cos u^2,\ 0\right)$$

Thus the metric components are $g_{ij} = X_i \bullet X_j$ so,

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix}$$

so

$$\det(g_{ij}) = \cos^2 u^1$$

which gives the usual formula for area

$$\mathrm{A}(X(\omega)) = \int_\omega \cos u^1\, du_1\, du_2.$$

We can consider the geometry on the surface and forget that the surface sits in three space. This is called the intrinsic geometry. Measurements like lengths and angles, computed form the Riemannian metric, are intrinsic quantities. If we endow an abstract two dimensional space M^2 with a Riemannian Metric, a smoothly varying inner product on each tangent space that is consistently defined on overlapping coordinate patches, the resulting object is an Abstract Riemannian Surface, or two dimensional Riemannian Manifold.

Mapping Question: Can the spherical surface be locally mapped to the Euclidean plane in such a way as to preserve the lengths of curves? What about an abstract Riemannian surface? What are the obstructions to doing this?

Let's view the Euclidean plane $\mathbb{E}^2$ as an abstract Riemannian surface. The space is $\mathbf{R^2}$ and the metric is

$$g_{ij}(v) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

The coordinate chart is $Y(v_1, u_2) = (v_1, u_2)$ and the tangent space is spanned by $Y_i = (1, 0)$ and $Y_2 = (0, 1)$. The inner product of two vector fields

$$\begin{aligned} A(v) &= a^1(v)Y_1(v) + a^2(v)Y_2(v), \\ B(v) &= b^1(v)Y_1(v) + b^2(v)Y_2(v) \end{aligned}$$

is

$$A \bullet B = \langle A, B\rangle = \sum_{i,j=1}^{2} \delta_{ij}\, a^i\, b^j = a^1 b^1 + a^2 b^2,$$

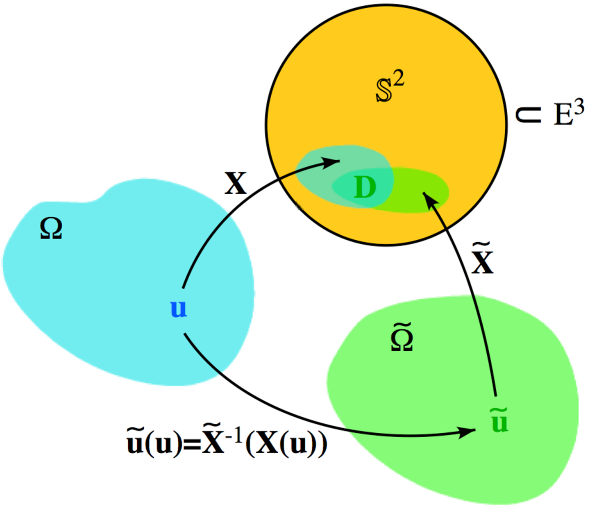
the usual Euclidean inner product of the plane.

What is obvious when we think of the spherical surface $\mathbb{S}^2 \subset \mathbf{R}^3$ is that regardless of what coordinate system we use in the neighborhood of $P \in \mathbb{S}^2$, the inner product between two vectors, or the area of a domain or the length of the curve is the same because they are expressions of the spherical values. *e.g.*, if we compute vectors and metrics in the Ω or the $\tilde{\Omega}$ coordinate systems near P, the inner product of the vector fields

$$\sum_{i,j=1}^{2} g_{ij}(u)\, a^i(u)\, b^j(u) = \langle A, B \rangle = \sum_{i,j=1}^{2} \tilde{g}_{ij}(\tilde{u})\, \tilde{a}^i(\tilde{u})\, \tilde{b}^j(\tilde{u}).$$

is the same where points $\tilde{u} = \tilde{u}(u)$ correspond on $\mathbb{S}^2$.

This also holds true in an abstract Riemanniam manifold. That is because the vector fields and the first fundamental form are tensors. Their transformations under change of coordinates exactly compensate to keep geometric quantities invariant under change of coordinate.



The geometric quantities in one chart may be computed in terms of the quantities in another.

Figure 4: Change of coordinate $\tilde{u}(u) = \tilde{X}^{-1}(X(u))$.

Suppose both $X : \Omega \to \mathbb{S}^2$ and $\tilde{X} : \tilde{\Omega} \to \mathbb{S}^2$ are two local coordinates and $P \in D = X(\Omega) \cap \tilde{X}(\tilde{\Omega})$. Then the change of variables is called a transition function. For $u \in X^{-1}(D)$ we have

$$X(u) = \tilde{X}(\tilde{u})$$

so $\tilde{u}(u) = (\tilde{X})^{-1}(X(u))$ whence by the chain rule

$$\frac{\partial X}{\partial u^p} = \sum_{i=1}^{2} \frac{\partial \tilde{X}}{\partial \tilde{u}^i} \frac{\partial \tilde{u}^i}{\partial u^p}$$

The formula for the tilde coordinates of a vector follow

$$A = \sum_{p=1}^{2} a^p \frac{\partial X}{\partial u^p} = \sum_{p=1}^{2} \sum_{i=1}^{2} a^p \frac{\partial \tilde{X}}{\partial \tilde{u}^i} \frac{\partial \tilde{u}^i}{\partial u^p} = \sum_{i=1}^{2} \left(\sum_{p=1}^{2} a^p \frac{\partial \tilde{u}^i}{\partial u^p} \right) \frac{\partial \tilde{X}}{\partial \tilde{u}^i}$$

yielding

$$\tilde{a}^i = \sum_{p=1}^{2} a^p \frac{\partial \tilde{u}^i}{\partial u^p}.$$

Similarly,

$$\tilde{g}_{ij}(\tilde{u}) = \sum_{k,\ell=1}^{2} g_{k\ell}(u(\tilde{u}))\frac{\partial u^k}{\partial \tilde{u}^i}\frac{\partial u^\ell}{\partial \tilde{u}^j}$$

The Jacobian matrices satisfy $\displaystyle\sum_{j=1}^{2}\frac{\partial \tilde{u}^i}{\partial u^j}\,\frac{\partial u^j}{\partial \tilde{u}^k} = \delta^i_k$ since they are inverses,

we can check that we get the same inner product in any coordinate system

$$\begin{aligned}
\langle A, B\rangle &= \sum_{i,j=1}^{2} \tilde{g}_{ij}(\tilde{u}(u))\,\tilde{a}^i(\tilde{u}(u))\,\tilde{b}^j(\tilde{u}(u)) \\
&= \sum_{i,j,k,\ell,p,q=1}^{2} g_{k\ell}\frac{\partial u^k}{\partial \tilde{u}^i}\frac{\partial u^\ell}{\partial \tilde{u}^j}a^p\frac{\partial \tilde{u}^i}{\partial u^p}b^q\frac{\partial \tilde{u}^j}{\partial u^q} \\
&= \sum_{p,q=1}^{2} g_{pq}(u)\,a^p(u)\,b^q(u).
\end{aligned}$$

Geometric quantities determined by the metric are called intrinsic. A map of the earth is a local diffeomorphism from the sphere to the Euclidean plane which is a local isometry that preserves lengths of curves, hence all intrinsic quantities. Equivalently, the Riemannian metrics are preserved. Thus if

$$f : (\mathbb{S}^2, g) \to (\mathbb{E}^2, \delta)$$

is an isometry, then f is a local diffeomorphism and $f^* \tilde{g} = \delta$ which means that for every vector fields A, B on $\mathbb{S}^2$ and at every point $P = X(u) \in \mathbb{S}^2$,

$$\langle A(u), B(u) \rangle_{\mathbb{S}^2} = \langle df_u(A(u)), df_u(B(u)) \rangle_{\mathbb{E}^2}$$

where $df_u : T_{X(u)}\mathbb{S}^2 \to T_{f(X(u))}\mathbb{E}^2$ is the differential.

WARNING: funtional analysts and geometric group theorists define "isometry" in a slightly different way.

If we denote $X : \Omega \to \mathbb{S}^2$ as a local coordinate of the sphere, $D = X(\Omega)$ and $v = (v^1, v^2) \in \mathbb{E}^2$ the standard Eclidean coordinate, then a map

$$f : D \to \mathbb{E}^2$$

may be written $v = f(X(u)) = (v^1(u^1, u^2),\ v^2(u^1, u^2))$. If $u(t) \in \Omega$ is a curve whose velocity vector in $\mathbb{S}^2$ is

$$A = \sum_{i=1}^{2} \frac{du^i}{dt} X_i$$

then $df(A)$ is the velocity vector of $f(X(u(t)))$, or

$$df(A) = \sum_{i,j=1}^{2} \frac{du^i}{dt} \frac{\partial v^j}{\partial u^i} Y_j$$

$$\langle A(u), B(u)\rangle_{\mathbb{S}^2} = \langle df_u(A(u)), df_u(B(u))\rangle_{\mathbb{E}^2}$$

becomes for all tangent vectors $A = \sum_{i=1}^{2} a^i X_i$ and $B = \sum_{j=1}^{2} a^i X_i$,

$$\sum_{i,j=1}^{2} g_{ij}(u)\, a^i\, b^j = \sum_{p,q=1}^{2} \delta_{pq} \left(\sum_{i=1}^{2} a^i \frac{\partial v^p}{\partial u^i}\right) \left(\sum_{j=1}^{2} b^j \frac{\partial v^q}{\partial u^j}\right)$$

As a^i and b^j were arbitrary, this reduces to the system for all i, j,

$$g_{ij}(u) = \sum_{p=1}^{2} \frac{\partial v^p}{\partial u^j}\, \frac{\partial v^p}{\partial u^j} = V_i \bullet V_j$$

for the unknown map $V = (v^1(u^1, u^2),\, v^2(u^1, u^2))$.

$$g_{ij} = V_i \bullet V_j \tag{2}$$

The assumption that a system of PDE's is soluble leads to conditions on the data, known as compatibility conditions. Differentiating

$$\frac{\partial g_{ij}}{\partial u^k} = V_{ik} \bullet V_j + V_i \bullet V_{jk}$$

where $V_{ij} = \dfrac{\partial^2 V}{\partial u^i \, \partial u^j}$. Cyclically permuting the indices and adding leaves

$$\begin{aligned} \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} &= V_{ij} \bullet V_k + V_i \bullet V_{kj} + V_{ki} \bullet V_j \\ &+ V_k \bullet V_{ji} - V_{ik} \bullet V_j - V_i \bullet V_{jk} \\ &= 2V_{ij} \bullet V_k \end{aligned}$$

since $V_{ij} = V_{ji}$.

$g_{ij} = V_i \bullet V_j$ implies that the map $V(u)$ has

$$\begin{aligned}
0 \neq \det(g_{ij}) &= g_{11}g_{22} - g_{12}^2 \\
&= [V_1 \bullet V_1]\,[V_2 \bullet V_2] - [V_1 \bullet V_2]^2 \\
&= \left[\left(\frac{\partial v^1}{\partial u^1}\right)^2 + \left(\frac{\partial v^2}{\partial u^1}\right)^2\right]\left[\left(\frac{\partial v^1}{\partial u^2}\right)^2 + \left(\frac{\partial v^2}{\partial u^2}\right)^2\right] \\
&\quad - \left[\frac{\partial v^1}{\partial u^1}\frac{\partial v^1}{\partial u^2} + \frac{\partial v^2}{\partial u^1}\frac{\partial v^2}{\partial u^2}\right]^2 \\
&= \left(\frac{\partial v^1}{\partial u^1}\frac{\partial v^2}{\partial u^2} - \frac{\partial v^2}{\partial u^1}\frac{\partial v^1}{\partial u^2}\right)^2 \\
&= \left|\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right|^2
\end{aligned}$$

nonvanishing Jacobian which means that the vectors V_i are linearly independent.

Thus the second derivative vectors may be written in terms of the basis

$$V_{ij} = \sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} \, V_\ell$$

where $\Gamma_{ij}^{k}(u)$ are coefficients depending on the second derivatives, called the Christoffel Symbols. We have shown that

$$\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} = 2V_{ij} \bullet V_k = 2\sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} \, V_\ell \bullet V_k = 2\sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} \, g_{\ell k}$$

Denoting the inverse matrix by $g^{km} = g^{km}$, we have multiplying both sides and summing

$$\frac{1}{2}\sum_{k=1}^{2} g^{mk} \left\{ \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right\} = \Gamma_{ij}^{m} \tag{3}$$

Thus the Christoffel symbols are determined by the metric and its first derivatives.

Using the fact that third derivatives commute, we obtain finally

$$
\begin{aligned}
0 = V_{ijk} - V_{ikj} &= \sum_{\ell=1}^{2} \left\{ \frac{\partial}{\partial u^k} \left(\Gamma_{ij}^{\ell} V_{\ell} \right) - \frac{\partial}{\partial u^j} \left(\Gamma_{ik}^{\ell} V_{\ell} \right) \right\} \\
&= \sum_{m=1}^{2} \left\{ \frac{\partial \Gamma_{ij}^{m}}{\partial u^k} - \frac{\partial \Gamma_{ik}^{m}}{\partial u^j} + \sum_{\ell=1}^{2} \left(\Gamma_{ij}^{\ell} \Gamma_{\ell k}^{m} - \Gamma_{ik}^{\ell} \Gamma_{\ell j}^{m} \right) \right\} V_m \\
&= \sum_{m=1}^{2} R_i{}^m{}_{jk} V_m
\end{aligned}
$$

The functions $R_i{}^m{}_{jk}$ are called the components of the Riemannian Curvature Tensor. Obviously $R_i{}^m{}_{jk} = -R_i{}^m{}_{kj}$.

This computation says that if there is an isometric mapping of the sphere to the plane, then the Riemannian Curvature of the metric must vanish.

Riemannian curvature is a tensor. This means that in any other curvilinear coordinate system, there is a transformation formula for the curvature tensor involving only multiplication by Jacobian matrices. It turns out that in a new coordinate system $\tilde{u}(u)$, the curvature is

$$\tilde{R}_i{}^m{}_{jk} = \sum_{p,q,r,s=1}^{2} R_p{}^q{}_{rs} \frac{\partial u^p}{\partial \tilde{u}^i} \frac{\partial \tilde{u}^m}{\partial u^q} \frac{\partial u^r}{\partial \tilde{u}^j} \frac{\partial u^s}{\partial \tilde{u}^k}$$

Thus the curvature is nonvanishing in any one coordinate system if and only if it is nonvanishing in any other coordinate system.

If we put $R_{i\ell jk} = \sum_{m=1}^{2} g_{m\ell} R_i{}^m{}_{jk}$ then some coefficients are equal

$$det(g_{pq})K = R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$$

and all the rest are zero. The scalar K is called the Gaussian Curvature of the surface. Thus curvature is called positive if $K > 0$.

The position and tangent vectors are

$$X = \begin{pmatrix} \cos u^1 \; \cos u^2 \\ \cos u^1 \; \sin u^2 \\ \sin u^1 \end{pmatrix} \; X_1 = \begin{pmatrix} -\sin u^1 \; \cos u^2 \\ -\sin u^1 \; \sin u^2 \\ \cos u^1 \end{pmatrix} \; X_2 = \begin{pmatrix} -\cos u^1 \; \sin u^2 \\ \cos u^1 \; \cos u^2 \\ 0 \end{pmatrix}$$

The metric components g_{ij} and its inverse are

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix} \quad \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sec^2 u^1 \end{pmatrix}$$

Hence

$$\frac{\partial g_{22}}{\partial u^1} = -2\cos u^1 \; \sin u^1 \quad \text{and} \quad \frac{\partial g_{ij}}{\partial u^k} = 0 \text{ for all other } i, j, k.$$

32. Christoffel Symbols of Spherical coordinates.

Christoffel symbols are $\Gamma^m_{ij} = \frac{1}{2}\sum_{k=1}^{2} g^{mk}\left\{\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right\}$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1}\right\} = 0$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1}\right\} = 0$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{21}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{22}}{\partial u^1}\right\} = \cos u^1 \ \sin u^1$$

$$\Gamma^2_{11} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\right\} = 0$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^2}\right\} = -\tan u^1$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^2}\right\} = 0$$

33. Curvature of Spherical Coordinates.

$$R_i{}^m{}_{jk} = \frac{\partial \Gamma^m_{ij}}{\partial u^k} - \frac{\partial \Gamma^m_{ik}}{\partial u^j} + \sum_{\ell=1}^{2} \left(\Gamma^\ell_{ij}\Gamma^m_{\ell k} - \Gamma^\ell_{ik}\Gamma^m_{\ell j} \right)$$

$$\begin{aligned} R_1{}^2{}_{12} &= \frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^2_{12}}{\partial u^1} + \sum_{\ell=1}^{2} \left(\Gamma^\ell_{11}\Gamma^2_{\ell 2} - \Gamma^\ell_{12}\Gamma^2_{\ell 1} \right) \\ &= 0 - \frac{\partial(-\tan u^1)}{\partial u^1} + 0 - (-\tan u^1)^2 \\ &= \sec^2 u^1 - \tan^2 u^1 = 1. \end{aligned}$$

$$K \det(g_{ij}) = K\cos^2 u^1 = R_{1212} = \sum_{m=1}^{2} g_{2m} R_1{}^m{}_{12} = \cos^2 u^1 \cdot 1$$

thus $K = 1$. The Gaussian Curvature of the sphere is identically one.

Since the curvature of the sphere does not vanish, it CANNOT BE LOCALLY ISOMETRICALLY MAPPED TO THE EUCLIDEAN PLANE.

It turns out that the vanishing of curvature is sufficient for the existence of a local isometric mapping of the surface to the Euclidean plane. Such surfaces are called flat and a local isometric mapping is called a development to the plane.

We are given a smooth metric g_{ij}, hence its Christoffel symbols Γ_{ij}^k, such that the curvatures vanish identically $R_i{}^m{}_{jk} = 0$. We seek an unknown local nondegenerate mapping $Z : \Omega \to \mathbb{E}^2$ that preserves the metric. As in higher order ODE's, we introduce new unknown variables Y_i that correspond to the derivatives of Z, and extend the system of differential equations. The resulting PDE is called a differential system.

$$\frac{\partial}{\partial u^i} Z = Y_i, \tag{4}$$

$$\frac{\partial}{\partial u^j} Y_i = \sum_{k=1}^{2} \Gamma ij^k \, Y_k, \tag{5}$$

$$Y_i \bullet Y_j = g_{ij} \tag{6}$$

Z has 2 and Y_i has 4 unknown functions. There 4 equations in (4), 8 in (5) and 3 in (6). All together there are 6 unknown functions satisfying 15 equations. The system is overdetermined.

An overdetermined system may be solved locally if there are no incompatibilities in the system. In this case, it amounts to checking whether the cross partial derivatives are consistent in equations (4) and (5) and the differentiated algebraic equations (6) are satisfied. Checking cross partials of (4) we have

$$\frac{\partial^2 Z}{\partial u^j \partial u^i} - \frac{\partial^2 Z}{\partial u^i \partial u^j} = \frac{\partial}{\partial u^j} Y_i - \frac{\partial}{\partial u^i} Y_j = \sum_{k=1}^{2} \left(\Gamma_{ij}^k - \Gamma_{ji}^k \right) Y_k = 0$$

because $\Gamma ij^k = \Gamma ji^k$.

Checking cross partials of (5) we have $\dfrac{\partial^2 Y_i}{\partial u^k \partial u^j} - \dfrac{\partial^2 Y_i}{\partial u^j \partial u^k}$

$$
\begin{aligned}
&= \frac{\partial}{\partial u^k}\left(\sum_{\ell=1}^{2} \Gamma_{ij}^{\ell} Y_\ell\right) - \frac{\partial}{\partial u^j}\left(\sum_{\ell=1}^{2} \Gamma_{ik}^{\ell} Y_\ell\right) \\
&= \sum_{m=1}^{2}\left(\frac{\partial}{\partial u^k}\Gamma_{ij}^{m} - \frac{\partial}{\partial u^j}\Gamma_{ik}^{m}\right) Y_m + \sum_{\ell=1}^{2}\left(\Gamma_{ij}^{\ell}\frac{\partial Y_\ell}{\partial u^k} - \Gamma_{ik}^{\ell}\frac{\partial Y_\ell}{\partial u^j}\right) \\
&= \sum_{m=1}^{2}\left\{\frac{\partial}{\partial u^k}\Gamma_{ij}^{m} - \frac{\partial}{\partial u^j}\Gamma_{ik}^{m} + \sum_{k=1}^{2}\left(\Gamma_{ij}^{k}\Gamma_{\ell k}^{m} - \Gamma_{ji}^{k}\Gamma_{kj}^{m}\right)\right\} Y_m \\
&= \sum_{m=1}^{2} R_i{}^m{}_{jk} Y_m
\end{aligned}
$$

using (5) and the assumption that $R_i{}^m{}_{jk}$ vanishes.

Checking derivatives of (6), we find

$$
\begin{aligned}
\frac{\partial}{\partial u^k}(Y_i \bullet Y_j) &= \frac{\partial Y_i}{\partial u^k} \bullet Y_j + Y_i \bullet \frac{\partial Y_j}{\partial u^k} \\
&= \left(\sum_{\ell=1}^{2} \Gamma_{ik}^{\ell} Y_\ell \right) \bullet Y_j + Y_i \bullet \left(\sum_{\ell=1}^{2} \Gamma_{jk}^{\ell} Y_\ell \right) \\
&= \sum_{\ell=1}^{2} \left(\Gamma_{ik}^{\ell} g_{\ell j} + \Gamma_{jk}^{\ell} g_{i\ell} \right) \\
&= \frac{1}{2} \left\{ \frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{ji}}{\partial u^k} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i} \right\} \\
&= \frac{\partial g_{ij}}{\partial u^k}
\end{aligned}
$$

as desired. We used (3) and the definition of Christoffel symbols (5).

Thus all compatibility conditions for the differential system hold: it can be solved locally to give a local isometry to Euclidean space.

Since the earth can't be mapped isometrically, we try to find maps that preserve some geometric feature. A map that preserves areas of regions is called equiareal.

There are many equiareal mappings such as those found by Albers, Collingnon and Gravé. Let us describe the projection of Lambert, which preserves the longitude lines. We seek a mapping of the form

$$v^1 = g(u^1) \qquad v^2 = u^2.$$

where, say, we have taken $(u^1, u^2) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi)$ to be the usual spherical coordinates latitude and longitude in radians. Thus $g_1 1 = 1$,$g_{12} = g_{21} = 0$ and $g_{22} = \cos^2 u^1$. Then the area forms must be equal

$$\begin{aligned} \det(g_{ij}) du^1\, du^2 &= \cos^2(u^1) du^1\, du^2 = dv^1\, dv^2 \\ &= \left| \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \right| du^1\, du^2 = \big(g'(u^1)\big)^2 du^1\, du^2 \end{aligned}$$

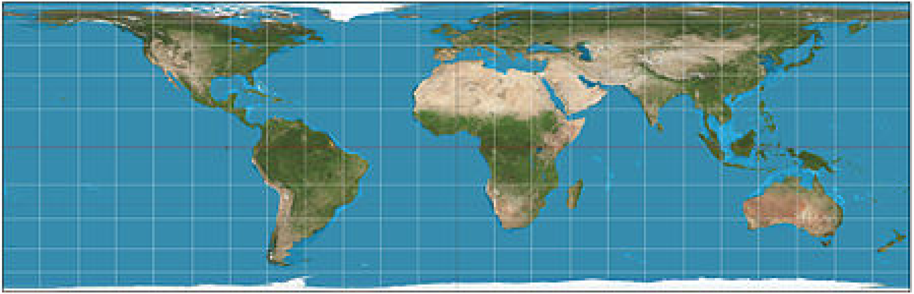


Figure 5: Lambert's Equiareal Projection

The differential equation for the Lambert projection is

$$g'(u^1) = \cos u^1$$

which for $g(0) = 0$ has the solution

$$v^1 = \sin u^1 \qquad v^2 = u^2.$$

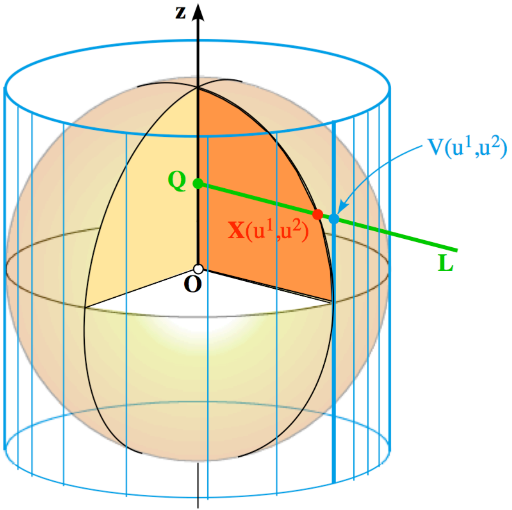


Figure 6: Projection from z-axis to Cylinder

Let $C = \{(\cos v^2, \sin v^2, v^1)\}$ be the circumscribed cylinder, a wrapped up Euclidean plane. For each $X(u^1, u^2) \in \mathbb{S}^2$, let the horizontal half line passing through X and the z-axis meets the cylinder at $V(u^1, u^2)$. $(u^1, u^2) \mapsto (v^1(u^1, u^2), v^2(u^1, u^2))$ is the equiareal Lambert Projection.

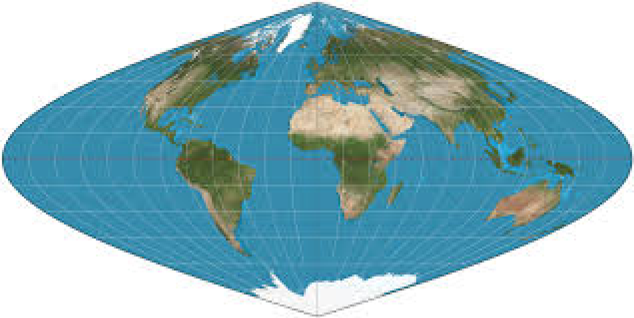


Figure 7: Sanson's Equiareal Projection

Sanson's (1650) projection is given by

$$v^1 = u^1 \qquad v^2 = u^2 \cos u^1.$$

Then the pulled back area form equals the spherical area form

$$dv^1\, dv^2 = du^1\,(-u^2 \sin u^1\, du^1 + \cos u^1\, du^2) = \cos u^1\, du^1\, du^2$$

Figure 8: Bonne Projection

Bonne's Projection is given
$r = \frac{\pi}{2} - u^1$
$\theta = \frac{u^2 \cos u^1}{u^1 - \frac{\pi}{2}}$.
where (r, θ) are polar coordinates of $\mathbb{E}^2$. The pulled back area is thus
$r\,dr\,d\theta = (\frac{\pi}{2} - u^1)\,du^1 \times \frac{(?)\,du^1 + \cos u^1\,du^2}{\frac{\pi}{2} - u^1}$
$= \cos u^1\,du^1\,du^2$.

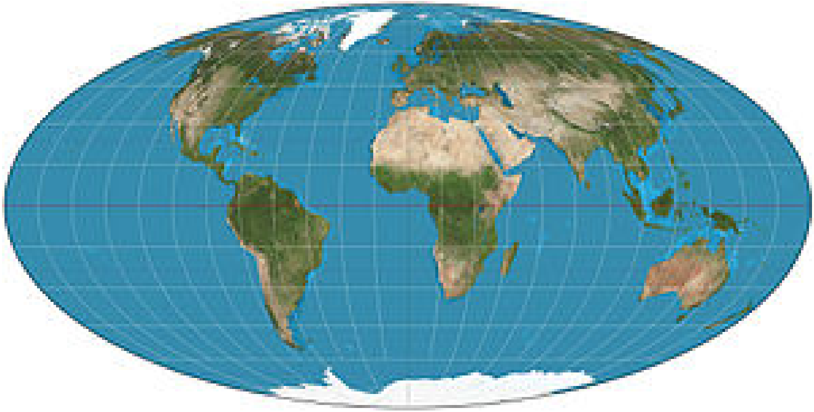


Figure 9: Mollweide's Equiareal Projection

The Mollweide equiareal projection is built of conics.

The angle between the vectors $A = a^1 X_1 + a^2 X^2$ and $B = b^1 X_1 + b^2 X^2$ is computed by the formula

$$\cos\alpha = \frac{\sum_{i,j=1}^{2} g_{ij} a^i \, b^j}{\sqrt{\sum_{i,j=1}^{2} g_{ij} a^i \, a^j}\sqrt{\sum_{i,j=1}^{2} g_{ij} b^i \, b^j}}$$

Thus if another metric satisfies $\tilde{g}_{ij} = \lambda g_{ij}$ where λ is a positive function, then $\tilde{g}_{ij}$ computes the same angle as does g_{ij} since the λ cancels. Such metrics are called conformal.
A mapping $v : \Omega \to \mathbb{E}^2$ is conformal if it induces a metric conformal to the spherical metric. It means that there is a function λ such that

$$\lambda(u^1, u^2) g_{ij}(u^1, u^2) = \sum_{k=1}^{2} \frac{\partial v^k}{\partial u^i} \frac{\partial v^k}{\partial u^j}$$

There are many conformal maps given by Lagrange and Von der Mühll and others.

45. Equations for Mercator's Conformal Map.

Let us describe the projection of mercator, which preserves the longitude lines. We seek a mapping of the form

$$v^1 = g(u^1) \qquad v^2 = u^2.$$

where, say, we have taken $(u^1, u^2) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi)$ to be the usual spherical coordinates latitude and longitude in radians. Thus
$g_1 1 = 1$,$g_{12} = g_{21} = 0$ and $g_{22} = \cos^2 u^1$.
We wish to find a $\lambda(u^1, u^2)$ such that

$$\begin{aligned}
\lambda g_{11} = \lambda &= \left(\frac{\partial v^1}{\partial u^1}\right)^2 + \left(\frac{\partial v^2}{\partial u^1}\right)^2 = \left(g'(u^1)\right)^2 \\
\lambda g_{12} = 0 &= \frac{\partial v^1}{\partial u^1}\frac{\partial v^1}{\partial u^2} + \frac{\partial v^2}{\partial u^1}\frac{\partial v^2}{\partial u^2} = 0 \\
\lambda g_{22} = \lambda \cos^2 u^1 &= \left(\frac{\partial v^1}{\partial u^2}\right)^2 + \left(\frac{\partial v^2}{\partial u^2}\right)^2 = 1
\end{aligned}$$

Figure 10: Mercator Projection

The third equation implies $\lambda = \sec^2 u^1$. thus, the differential equation for the Mercator projection is

$$g'(u^1) = \sec u^1$$

which for $g(0) = 0$ has the solution

$$\begin{aligned} v^1 &= \log\left(\sec u^1 + \tan u^1\right) \\ &= \log\tan\left(\frac{\pi}{4} + \frac{u^1}{2}\right) \\ v^2 &= u^2. \end{aligned}$$

47. Mercator Projection.

Figure 11: Political Mercator Map

Gerhard Kremer, known as Mercator (1512-1594), first published a map using his projection in 1569.

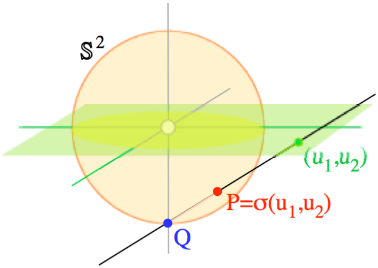


Figure 12: Stereographic Projection. $P = \sigma(u_1, u_2)$ is the point on the sphere corresponding to $(u_1, u_2) \in \mathbf{R}^2$.

This is another coordinate chart for the unit sphere $\sigma : \mathbf{R}^2 \to \mathbb{S}^2$

For the unit sphere $\mathbb{S}^2$ centered at the origin, imagine a line through the south pole Q and some other point $P \in \mathbb{S}^2$. This line crosses the $z = 0$ plane at some coordinate $x = u_1$ and $y = u_2$. Then we can express P in terms of (u_1, u_2). Thus $\sigma : U = \mathbf{R}^2 \to \mathbb{S}^2 - \{Q\}$ is a coordinate chart for the sphere called stereographic coordinates.

$$\sigma(u_1, u_2) = \left(\frac{2u_1}{1+u_1^2+u_2^2}, \frac{2u_2}{1+u_1^2+u_2^2}, \frac{1-u_1^2-u_2^2}{1+u_1^2+u_2^2}\right)$$

The tangent vectors for stereographic projection are

$$\sigma_1 = \left(\frac{2-2u_1^2+2u_2^2}{\left(1+u_1^2+u_2^2\right)^2}, -\frac{4u_1u_2}{\left(1+u_1^2+u_2^2\right)^2}, -\frac{4u_1}{\left(1+u_1^2+u_2^2\right)^2} \right),$$
$$\sigma_2 = \left(-\frac{4u_1u_2}{\left(1+u_1^2+u_2^2\right)^2}, \frac{2+2u_1^2-2u_2^2}{\left(1+u_1^2+u_2^2\right)^2}, -\frac{4u_2}{\left(1+u_1^2+u_2^2\right)^2} \right)$$

so that the metric satisfies

$$\sigma_1 \bullet \sigma_1 = \frac{4}{\left(1+u_1^2+u_2^2\right)^2}, \qquad \sigma_1 \bullet \sigma_2 = 0, \quad \sigma_2 \bullet \sigma_2 = \frac{4}{\left(1+u_1^2+u_2^2\right)^2}$$

Thus stereographic coordinates already yield a conformal mapping because $g_{ij} = \lambda(u^1, u^2)\,\delta_{ij}$ is a scalar multiple of the Euclidean metric.

Figure 13: Stereographic Projection is Conformal

Stereographic coordinates are given by $\sigma(u_1, u_2) =$

$$\begin{pmatrix} \dfrac{2u_1}{1+u_1^2+u_2^2} \\[2ex] \dfrac{2u_2}{1+u_1^2+u_2^2} \\[2ex] \dfrac{1-u_1^2-u_2^2}{1+u_1^2+u_2^2} \end{pmatrix}$$

Since the surface of the earth cannot be isometrically mapped to the plane, the interesting mathematical question arises: what is the least distortion mapping of a given region $D \subset \mathbb{S}^2$?

Let $d_S(P, Q)$ denote the geodesic distance between the points P and Q of $\mathbb{S}^2$. This is equal to the length of the shorter great circle arc joining P to Q. The Euclidean distance between two points V and W in the plane will be denoted by $d_E(V, W)$.

Let $D \subset \mathbb{S}^2$ and $f : D \to \mathbb{E}^2$ be a mapping The scale of map distortion between two points $P, Q \in D$ is defined to be the ratio

$$\frac{d_E(f(P), f(Q))}{d_S(P, Q)}$$

Ideally we want this to be constant but this may not be possible. If we let σ_1 to be the infimum of this ratio as $P \neq Q$ vary through all points of D and σ_2 the supremum, then these are the best constants such that the inequality holds for all $P, Q \in D$

$$\sigma_1 d_S(P, Q) \le d_E(f(P), f(Q)) \le \sigma_2 d_S(P, Q)$$

Milnor defines logarithm of max to min scales

$$\delta = \log(\sigma_2/\sigma_1)$$

as the distortion of the mapping f of D.

Let D_α denote a spherical cap, the region bounded by a circle of the sphere. Given a fixed point $P_0 \in \mathbb{S}^2$ and $0 < \alpha < \pi$, the set of points $P \in \mathbb{S}^2$ that are a geodesic distance

$$D_\alpha(P_0) = \{P \in \mathbb{S}^2 : d_S(P, P_0) \leq \alpha\}$$

Theorem (Milnor, 1969)

There is one, and up to similarity transformations of the plane, only one minimum distotrtion map $f : D_\alpha(P_0) \to \mathbb{E}^2$. It is given by the azimuthal equidistant projection from P_0 and has

$$\delta = \log(\alpha/\sin\alpha).$$

The azimuthal equidistant projection preserves distances from P_0 and angles at P_0. If we let $P_0 = (0, 0, 1)$ be the north pole and write the sphere in polar cooordinates

$$X(u^1, u^2) = (\sin u^1 \, \cos u^2, \sin u^1 \, \sin u^2, \cos u^1)$$

where u^1 is the distance from the north pole and u^2 the longitude. The tangent vectors are

$$X_1(u^1, u^2) = \left(\cos u^1 \, \cos u^2, \; \cos u^1 \, \sin u^2, \; -\sin u^1\right),$$
$$X_2(u^1, u^2) = \left(-\sin u^1 \, \sin u^2, \; \sin u^1 \, \cos u^2, \; 0\right)$$

The metric is $g_{ij} = X_i \bullet X_j$ so $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u^1 \end{pmatrix}$
The Azimuthal Equidistant Projection is polar coordinates of the plane

$$v^1 = u^1 \, \cos u^2, \qquad v^2 = u^1 \sin u^2$$

Figure 14: Azimuthal Equidistant Projection

The Azimuthal Equidistant Projection is polar coordinates u^1 =distance from pole and u^2 =angle at pole of a spherical cap mapped to polar coordinates u^1 =distance from origin and u^2 =angle at origin in the plane.

$$v^1 = u^1 \cos u^2,$$
$$v^2 = u^1 \sin u^2$$

Lemma (1.)

The distortion δ of any map $f : D_\alpha(P_0) \to \mathbb{E}^2$ satisfies $\delta \geq \log(\alpha/\sin\alpha)$.

Proof Part 1.

We may assume f has finite distortion. Hence the Lipschitz inequalities

$$\sigma_1 d_S(P,Q) \leq d_E(f(P), f(Q)) \leq \sigma_2 d_S(P,Q)$$

hold. The upper one shows that f is continuous. The lower shows it is one-to-one. Let C_α denote the bounding circle of D_α. Its image $f(C_\alpha)$ is a simple closed curve in the plane.

First we show that every half-line emanating from $f(P_0)$ in $\mathbb{E}^2$ must intersect $f(C_\alpha)$ at least once. By the Jordan Curve Theorem, the curve cuts the plane into two disjoint components $\mathbb{E}^2 - f(C_\alpha) = E_1 \cup E_2$ where one, E_1, is bounded and must be the image of the interior of D_α and contain $f(P_0)$. But the half lines are unbounded and cannot be entirely contained in E_1, thus must cross $f(C_\alpha)$, proving the first assertion.

Proof Part 2.

By the Lipschitz bound, $f(C_\alpha)$ has finite length

$$L \leq 2\pi\sigma_2 \sin\alpha. \tag{7}$$

Since every point of $Q_0 \in C_\alpha$ has geodesic distance $d_S(P_0, Q_0) = \alpha$, it follows from the lower bound that every point

$$\sigma_1\alpha \leq d_E(f(P_0), f(Q_0)).$$

Thus the curve $f(C_\alpha)$ lies outside the disk D^* centered at $f(P_0)$ and radius $\sigma_1\alpha$.

Second we show that $L \geq 2\pi\sigma_1\alpha$, with equality if and only if $f(C_\alpha) = \partial D^*$. □

Proof Part 3.

Cut $f(C_\alpha)$ by a line through the disk center $f(P_0)$ and choose two line points $A, B \in f(C_\alpha)$ on opposite sides of D^*. Let γ be one of the two arcs of $f(C_\alpha)$ from A to B. Using polar coordinates centered at P_0, we may assume that the arc γ is given by piecewise differentiable functions

$$\rho = \rho(t), \qquad \theta = \theta(t), \qquad \text{for } 0 \leq t \leq 1.$$

It follows from $\rho \geq \sigma_1 \alpha$ that

$$\mathrm{L}(\gamma) = \int_0^1 \sqrt{\dot{\rho}^2 + \rho^2 \dot{\theta}^2}\, dt \geq \int_0^1 \rho |\dot{\theta}|\, dt \geq \sigma_1 \alpha \left| \int_0^1 \dot{\theta}\, dt \right| \geq \sigma_1 \alpha \pi.$$

Adding both arc lengths, we find $\mathrm{L}(\gamma) \geq 2\pi\sigma_1\alpha$.
If γ were not piecewise differentiable, the same length inequality could be obtained by an approximation argument.

□

Proof Part 4.

Now suppose that $\mathsf{L}(\gamma) = \alpha\pi\sigma_1$. If any portion of γ is strictly outside D^*, then a small subarc may be removed from γ, replaced by a straight line segment outside D^* with shorter length, which is impossible.
It follows that $\mathsf{L}(\gamma) = \alpha\pi\sigma_1$ implies that $f(C_\alpha)$ is the cull circle ∂D^*.

Combining with the second assertion with (7) we obtain

$$2\pi\sigma_1\alpha \leq 2\pi\sigma_2 \sin\alpha$$

from which it follows that

$$\frac{\alpha}{\sin\alpha} \leq \frac{\sigma_2}{\sigma_1}.$$

and hence $\log(\alpha/\sin\alpha) \leq \delta$. □

Lemma (2.)

Let $f : D_\alpha(P_0) \to \mathbb{E}^2$ *be a map with distortion* $\delta = \log(\alpha/\sin\alpha)$. *Then* f *is an azimuthal equidistant projection.*

Proof Part 1.

We have already shown that if $\delta = \log(\alpha/\sin\alpha)$ then $f(C_\alpha)$ is the circle centered at $f(P_0)$ of radius $\sigma_1\alpha = \sigma_2 \sin\alpha$ and $f(D_\alpha)$ is the disk D^* bounded by this circle.

Consider a point $Q \in D_\alpha$. Construct a great circle arc from P_0 through Q to $\bar{Q} \in C_\alpha$. If $c = d_S(P_0, Q)$ is the geodesic distance, then $d_S(Q, \bar{Q}) = \alpha - c$ and the distance to every other point on C_α is strictly greater. By the Lipschitz bound, $d_E(f(P_0, f(Q)) \geq \sigma_1 c$, $d_E(f(Q), f(\bar{Q})) \geq \sigma_1(c - \alpha)$ and $d_E(f(Q), f(Q')) > \sigma_1(c - \alpha)$ for every other $Q' \in C_\alpha - \bar{Q}$. These three conditions hold for exactly one point in D^*, namely the point at distance $\sigma_1 c$ on the line segment from $f(P_0)$ to $f(\bar{Q})$. Thus the map f is completely determined by what it does to the points of C_α.

□

Proof Part 2.

It remains to show that the the circle C_α is carried to the circle $f(C_\alpha)$ by a similarity transform that preserves angles between points. Cut C_α into two arcs γ and γ' such that

$$\mathsf{L}(\gamma) + \mathsf{L}(\gamma') = \mathsf{L}(C_\alpha) = 2\pi \sin\alpha$$

By the Lipschitz inequality

$$\mathsf{L}(f(\gamma)) \le \sigma_2\, \mathsf{L}(\gamma), \qquad \mathsf{L}(f(\gamma')) \le \sigma_2\, \mathsf{L}(\gamma') \tag{8}$$

But

$$\mathsf{L}(f(\gamma)) + \mathsf{L}(f(\gamma')) = \mathsf{L}(f(C_\alpha)) = 2\pi\sigma_2 \sin\alpha.$$

Hence the inequalities in (8) are equalities, proving the lemma.

□

Lemma (3.)

The azimuthal equidistant projection $f : D_\alpha \to \mathbb{E}^2$ has distortion $\delta = \log(\alpha/\sin\alpha)$.

Proof. Part 1.

For simplicity, put $f(P_0) = 0$ and use $\rho =$distance to north pole P_0 and $\theta =$longitude instead of (u^1, u^2). Then the metric of the sphere and the metric at the corresponding points in $\mathbb{E}^2$ are

$$ds_S^2 = d\rho^2 + \sin^2\rho\, d\theta^2; \qquad ds_E^2 = d\rho^2 + \rho^2\, d\theta^2$$

For a piecewise smooth curve $(\rho(t), \theta(t))$ in D_α and for the corresponding curve in $f(D_\alpha)$ we have

$$L_S = \int \sqrt{\dot\rho^2 + \sin^2\rho\,\dot\theta^2}\, dt \qquad L_E = \int \sqrt{\dot\rho^2 + \rho^2\,\dot\theta^2}\, dt.$$

□

Proof. Part 2.

Since $\rho/\sin\rho$ is a monotone increasing function of ρ we have

$$\sin\rho \leq \rho \leq \frac{\alpha}{\sin\alpha}\sin\rho$$

from which it follows that

$$L_S \leq L_E \leq \frac{\alpha}{\sin\alpha} L_S. \tag{9}$$

From this we'll deduce that for every $P, Q \in D_\alpha$

$$d_S(P,Q) \leq d_E(f(P), f(Q)) \leq \frac{\alpha}{\sin\alpha} d_S(P,Q).$$

which implies $\delta \leq \log(\alpha/\sin\alpha)$, and by Lemma 1, that $\delta = \log(\alpha/\sin\alpha)$.

$\square$

Proof. Part 3.

The main fact is that the distance between two points the infimum of the lengths of all piecewise smooth curves connecting the points. To see the first inequality, join $f(P)$ to $f(Q)$ in the convex set $f(D_\alpha)$ by a line segment which realizes the minimum length $L_E = d(f(P), f(Q))$. The corresponding curve in D_α will have length $L_S \geq d_S(P.Q)$. Since $L_S \leq L_E$ we conclude $d_S(P, Q) \leq d_E(P, Q)$.

If $\alpha \leq \pi/2$ the same argument works because D_α is geodesically convex. The great circle arc joining P to Q that minimizes the length remains in D_α and has length $L_S = d_S(P, Q)$. Its image curve has length $L_E \leq \frac{\alpha}{\sin \alpha} L_S$ do that $d_E(P, Q) \leq \frac{\alpha}{\sin \alpha} d_S(P, Q)$.

If $\alpha > \pi/2$, it may happen that shortest connecting curve, the great circle arc leaves D_α. That is it passes through a boundary point at P' and then crosses back in at another boundary point Q'. □

Proof. Part 4.

We show that

$$d_E(f(P), f(P')) \leq \tfrac{\alpha}{\sin\alpha} d_S(P, P') \tag{10}$$

$$d_E(f(P'), f(Q')) \leq \tfrac{\alpha}{\sin\alpha} d_S(P', Q') \tag{11}$$

$$d_E f(Q'), f(Q)) \leq \tfrac{\alpha}{\sin\alpha} d_S(Q', Q) \tag{12}$$

The sum of the left sides exceeds $d_S(P, Q)$ by the triangle inequality and the sum of the right sides equals $\frac{\alpha}{\sin\alpha} d_S(P, Q)$ since they are all on the same great circle arc, proving the required inequality.

Inequalities (10) and (12) can be proved by the argument above.

□

Proof. Part 5.

To see (11), consider the azimuthal equidistant projection g from the south pole P_0', whose domain is the complementary disk $D_{\pi-\alpha}(P_0')$. Observe that the points $g(P')$ and $g(Q')$ are on the circle of radius $\pi-\alpha$ about $g(P_0')$ with the same angle $\angle f(P')0f(Q') = \angle g(P')g(P_0')g(Q')$. Now since $\pi-\alpha<\pi/2$, we have as before,

$$\frac{\pi-\alpha}{\alpha} d_E(f(P'),f(Q')) = d_E(g(P'),g(Q') \le \frac{\pi-\alpha}{\sin(\pi-\alpha)} d_S(P',Q')$$

Multiplying by $\alpha/(\pi-\alpha)$ yields the inequality (11). This completes the proof of the lemma. □

Proof of Milnor's Theorem.

Lemmas 1, 2 and 3 imply Milnor's Theorem. □

If $D \subset \mathbb{S}^2$ is an open set and $f : D \to \mathbb{S}^2$ is conformal, then f has a well defined *infinitesimal scale*

$$\sigma(P) = \lim_{Q \to P} \frac{d_E(f(Q), f(P))}{d_S(Q, P)}$$

Theorem (P. L. Chebychef's Theorem (1860))

If D is a simply connected region bounded by a twice differentiable curve, then there exists one, and up to similarity transformations of $\mathbb{E}^2$, only one conformal map projection which minimizes

$$\frac{\sup_{P \in D} \sigma(P)}{\inf_{P \in D} \sigma(P)}.$$

This "best possible" conformal mapping is characterized by property that its infinitesimal scale function σ is constant along the boundary of D.

Thanks!