Math 6410 § 1. **First Midterm Exam** Name: Solutions
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1. *Let $f(t,x)$ be continuous for all $(t,x) \in \mathbf{R} \times \mathbf{R}^d$ and let $x_0 \in \mathbf{R}^d$. Suppose that there is $L \in \mathbf{R}$ such that*

$$|f(t,x) - f(t,y)| \le L|x-y|, \qquad \text{for all } (t,x),(t,y) \in \mathbf{R} \times \mathbf{R}^d. \tag{1}$$

Show that there is and $\varepsilon > 0$ and a unique function $y \in \mathcal{C}^1([0,\varepsilon], \mathbf{R}^d)$ that satisfies the initial value problem. Do not just quote theorems. Provide as detailed a proof as you can.

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \\ x(0) = x_0. \end{cases} \tag{2}$$

Finding a solution of (2) is equivalent to finding a continuous solution of the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds \qquad \text{for } t \in [0,\varepsilon]. \tag{3}$$

If $x(t)$ is a $\mathcal{C}^1$ solution of (2) then by the Fundamental Theorem of Calculus

$$x(t) = x(0) + \int_0^t \frac{dx}{dt}(s)\, ds = x_0 + \int_0^t f(s,x(s))\, ds.$$

On the other hand, if $x(t)$ is a continuous solution of (3), then since the function $s \mapsto f(s,x(s))$ is continuous, then by the other Fundamental Theorem of Calculus, the right side of (3) is differentiable, and its derivative is

$$\frac{dx}{dt}(t) = 0 + f(t, x(t)),$$

which is continuous. Hence $x(t)$ is a $\mathcal{C}^1$ solution of the differential equation of (2). Moreover, since $x(0) = x_0 + 0$ in (3), $x(t)$ also satisfies the initial condition in (2).

Solubility of the integral equation shall follow from the Contraction Mapping Theorem.

Theorem 1 (Contraction Mapping Theorem)**.** *Let X be a Banach Space (complete normed linear space) with norm $\|\bullet\|$. Suppose $\mathcal{L} : X \to X$ be a mapping which is a contraction: there is a $\theta \in (0,1)$ such that*

$$\|\mathcal{L}[x] - \mathcal{L}[y]\| \le \theta \|x-y\|, \qquad \text{for all } x,y \in X. \tag{4}$$

Then there is a unique fixed point $z \in X$ such that $\mathcal{L}[z] = z$.

Proof. Choose any point $z_0 \in X$ and consider the iteration sequence defined recursively by $z_1 = \mathcal{L}[z_0]$ and $z_{n+1} = \mathcal{L}[z_n]$ for $n = 1, 2, 3, \ldots$. Using (4), we see by induction that

$$\begin{aligned} \|z_2 - z_1\| &= \|\mathcal{L}[z_1] - \mathcal{L}[z_0]\| \le \theta\|x_1 - z_0\| \\ \|z_3 - z_2\| &= \|\mathcal{L}[z_2] - \mathcal{L}[z_1]\| \le \theta\|x_2 - z_1\| \le \theta^2\|x_1 - z_0\| \\ &\vdots \\ \|z_{n+1} - z_n\| &= \|\mathcal{L}[z_n] - \mathcal{L}[z_{n-1}]\| \le \theta\|x_n - z_{n-1}\| \le \theta^n\|x_1 - z_0\| \end{aligned} \tag{5}$$

We show $\{z_n\}$ is a Cauchy Sequence in X. Choose $\eta > 0$. Let $N \in \mathbf{R}$ be so large that

$$\frac{\theta^N}{1-\theta}\|z_1 - z_0\| < \eta.$$

Then for any $p, q \in \mathbb{N}$ such that $p > N$ and $q > N$, either $p = q$ in which case $\|z_p - z_q\| = 0 < \eta$ or, because $\|z_p - z_q\| = \|z_q - z_p\|$ we may suppose, after swapping values if necessary, that $p > q$. Then using a telescoping sum, the triangle inequality, (5) and the geometric sum formula,

$$\begin{aligned}
\|z_p - z_q\| &= \|(z_p - z_{p-1}) + (z_{p-1} - z_{p-2}) + \cdots + (z_{q+1} - z_q)\| \\
&\le \|z_p - z_{p-1}\| + \|z_{p-1} - z_{p-2}\| + \cdots + \|z_{q+1} - z_q)\| \\
&\le \theta^{p-1}\|z_1 - z_0\| + \theta^{p-2}\|z_1 - z_0\| + \cdots + \theta^q\|z_1 - z_0\| \\
&\le \theta^q \left(\theta^{p-q-1} + \theta^{p-q-2} + \cdots + \theta + 1\right)\|z_1 - z_0\| \\
&= \theta^q \left(\frac{1 - \theta^{p-q}}{1 - \theta}\right)\|z_1 - z_0\| \le \frac{\theta^q}{1-\theta}\|z_1 - z_0\| < \frac{\theta^N}{1-\theta}\|z_1 - z_0\| < \eta
\end{aligned}$$

because $q > N$. Hence $\{z_n\}$ is a Cauchy Sequence.

Because X is complete, there is a limit point $z \in X$ such that

$$z = \lim_{n\to\infty} z_n.$$

By continuity (4) we may exchange $\mathcal{L}$ with the limit

$$z = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} \mathcal{L}[z_n] = \mathcal{L}\left[\lim_{n\to\infty} \ddagger_n\right] = \mathcal{L}[z],$$

so z is a fixed point. The fixed point is unique. If there were another, $y = \mathcal{L}[y]$, then

$$\|y - z\| = \|\mathcal{L}[y] - \mathcal{L}[z]\| \le \theta\|y - z\|$$

so

$$(1 - \theta)\|y - z\| \le 0$$

implying $\|y - z\| = 0$ so $y = z$ since $1 - \theta > 0$. Thus there is only one fixed point in X. □

Choose any positive $\varepsilon < 1/L$ and let $I = [0, \varepsilon]$. Define the Banach Space of continuous function $X = \mathcal{C}(I, \mathbf{R}^d)$ with sup norm

$$\|f\| := \sup_{t\in I} |f(t)|.$$

Define the Picard operator for $f \in X$ and $t \in I$ by

$$\mathcal{L}[f](t) = x_0 + \int_0^t f(s, x(s))\, ds.$$

We seek a fixed point $x \in X$ of the Picard operator $x = \mathcal{L}[x]$ which is a solution of the integral equation (3).

$\mathcal{L}$ maps X to X. To see this, for any $x \in X$, since $s \mapsto f(s, x(s))$ is continuous, then by the Fundamental Theorem of Calculus the indefinite integral $\mathcal{L}[x]$ is continuous, so in X.

Finally, $\mathcal{L}$ is a contraction on X. Choose $x, y \in X$. Then for $t \in I$, using (1),

$$\begin{aligned}
|\mathcal{L}[x](t) - \mathcal{L}[y](t)| &= \left| x_0 + \int_0^t f(s, x(s))\, ds - x_0 - \int_0^t f(s, y(s))\, ds \right| \\
&= \left| \int_0^t f(s, x(s)) - f(s, y(s))\, ds \right| \\
&\le \int_0^t \big| f(s, x(s)) - f(s, y(s)) \big|\, ds \\
&\le \int_0^t L \big| x(s) - y(s) \big|\, ds \\
&\le \int_0^t L \|x - y\|\, ds \\
&= Lt\|x - y\| \le L\varepsilon\|x - y\|.
\end{aligned}$$

Taking supremum over $t \in I$,

$$\|\mathcal{L}[x] - \mathcal{L}[y]\| \le L\varepsilon\|x - y\|.$$

Thus $\mathcal{L}$ is a contraction in X with constant $\theta = L\varepsilon < 1$.

By the Contraction Mapping Theorem, there is a unique $x \in X$ which is a fixed point of $\mathcal{L}$, hence a solution of the integral equation and the Initial Value Probelm (2).

2. *Consider Griffith's model for a genetic control system, where x and y are proportional to the concentration of a protein and the messenger RNA from which it is translated, respectively, and $\mu > 0$ is a rate constant. Let $x_0 > 0$ and $y_0 > 0$.*

$$\begin{aligned}
\dot{x} &= y - \mu x & x(0) &= x_0 \\
\dot{y} &= \frac{x^2}{1 + x^2} - y. & y(0) &= y_0
\end{aligned} \tag{6}$$

Assume that the solution of (6) exists for all $t \in [0, T]$. Find a bound on the solution at time t. Uning your bound, explain why the solution of (6) exists for all $t \in [0, \infty)$.

[You may use theorems, but state carefully any theorem that you use and verify that the hypotheses hold.]

Let us write (6) as a vector equation.

$$\begin{cases} \dfrac{dz}{dt} = Az + b(z) = f(z, \mu), \\ \qquad x(0) = z_0. \end{cases} \tag{7}$$

where

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad A = \begin{pmatrix} -\mu & 1 \\ 0 & -1 \end{pmatrix}, \quad b(z) = \begin{pmatrix} 0 \\ x^2/(1 + x^2) \end{pmatrix}$$

$f(z, \mu)$ is a rational function so it is $\mathcal{C}^k$ in (z, μ) for all k. We estimate the solution using Gronwall's Inequality.

Theorem 2 (Gronwall's Inequality)**.** *Let α, β be real constants such that $\beta \ge 0$. If $u(t)$ is a continuous function satisfying*

$$u(t) \le \alpha + \beta \int_0^t u(s)\, ds, \qquad \textit{for all } t \in [0, T], \tag{8}$$

. *then*

$$u(t) \le \alpha e^{\beta t} \qquad \text{for all } t \in [0, T]. \tag{9}$$

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Estimating the operator norm and length, for all $z \in \mathbf{R}^2$,

$$\|A\| \le \|A\|_2 = \sqrt{\mu^2 + 2}, \qquad |b(z)| \le 1.$$

Estimating the integral equation, for $t \in [0, T]$,

$$\begin{aligned} |z(t)| &= \left| z_0 + \int_0^t Az(s) + b(z(s))\, ds \right| \\ &\le |z_0| + \int_0^t |Az(s) + b(z(s))|\, ds \\ &\le |z_0| + \int_0^t |Az(s)| + |b(z(s))|\, ds \\ &\le |z_0| + \int_0^t \|A\|\, |z(s)| + 1\, ds + \\ &\le |z_0| + t + \|A\| \int_0^t |z(s)|\, ds \\ &\le |z_0| + T + \|A\| \int_0^t |z(s)|\, ds \end{aligned}$$

Thus applying Gronwall's Inequality to $u(t) = |z(t)|$, $\alpha = |u_0| + T$ and $\beta = \|A\| = \sqrt{\mu^2 + 2}$ we have for all $t \in [0, T]$,

$$|z(t)| \le (|z_0| + T)e^{t\sqrt{\mu^2+2}}. \tag{10}$$

To see that the solution exists on all of $t \in [0, \infty)$, we invoke the global existence theorem.

Theorem 3 (Global Existence Theorem)**.** *Let $f(x, \mu) : \mathbf{R}^d \times \mathbf{R}^p \to \mathbf{R}^d$ be a $\mathcal{C}^1$ function. Then for every $(x_0, \mu) \in \mathbf{R} \times \mathbf{R}^d \times$, the initial value problem*

$$\begin{cases} \dfrac{dz}{dt} = f(z, \mu), \\ x(0) = z_0. \end{cases}$$

has a unique maximal $\mathcal{C}^1$ solution $\gamma(t)$ defined on $I_{x_0,\mu} = (a, b)$ its maximal interval of existence. If $b < \infty$, then $|\gamma(t)| \to \infty$ as $t \to b-$.

The right side $f(z, \mu)$ of (7) is $\mathcal{C}^1$ so we may apply the Global Existence Theorem. Let $\gamma(t)$ be the maximal solution of (7) and suppose for contradiction that it does not extend to infinity so $b < \infty$. Then by the Global existence theorem, the trajectory exits any compact set, *i.e.,* there is a time $t_0 \le T < b$ such that

$$|\gamma(T)| > (|z_0| + b)e^{b\sqrt{\mu^2+2}}.$$

But this contradicts the estimate (10) we obtained at $t = T$, namely the solution existing on $[t_0, T]$ satisfies

$$|\gamma(t)| \le (|z_0| + T)e^{t\sqrt{\mu^2+2}} < (|z_0| + b)e^{b\sqrt{\mu^2+2}}$$

for all $t \in [t_0, T]$. Thus, b is not finite and the solution $\gamma(t)$ exists for all $t \in [t_0, \infty)$.

3. *Consider again the ODE from Problem 2, where $\mu > 0$ is a constant.*

$$\begin{aligned} \dot{x} &= y - \mu x & x(0) &= x_0 \\ \dot{y} &= \frac{x^2}{1+x^2} - y. & y(0) &= y_0 \end{aligned} \tag{11}$$

Let $x_0 > 0$ and $y_0 > 0$. Explain why the solution may be differentiated with respect to μ. How big are $\frac{\partial x}{\partial \mu}(T,\mu)$ and $\frac{\partial y}{\partial \mu}(T,\mu)$ at $T > 0$?

The differentiability of the solution with respect to parameters follows from the Global Differentiable Dependence Theorem.

Theorem 4 (Global Differentiable Dependence Theorem)**.** *Let $f(x,\mu) : \mathbf{R}^d \times \mathbf{R}^p \to \mathbf{R}^d$ be a $\mathcal{C}^k$ function. Then the map $\mathcal{D} \to \mathbf{R}^d$ given by the solution $(t, \bar{z}, \bar{\mu}) \mapsto z(t, \bar{x}, \bar{\mu})$ of*

$$\begin{cases} \dfrac{dz}{dt} = f(z, \bar{\mu}), \\ \quad z(0) = \bar{z}. \end{cases} \tag{12}$$

is $\mathcal{C}^k$ in the domain

$$\mathcal{D} = \left\{ (t, \bar{z}, \bar{\mu}) \in \mathbf{R}^d \times \mathbf{R}^p : t \in I_{\bar{z},\bar{\mu}} \right\}.$$

where $I_{\bar{x},\bar{\mu}}$ is the maximal interval of existence for the solution of (7). Moreover, the function

$$W(t) = D_\mu z(t; \bar{x}, \bar{\mu})$$

satisfies the variational equation

$$\begin{cases} \dfrac{dW}{dt} = D_z f(z(t, \bar{z}, \bar{\mu}), \bar{\mu})\, W(t) + D_\mu f(z(t, \bar{z}, \bar{\mu}), \\ \qquad\qquad W(0) = 0. \end{cases} \tag{13}$$

The right side of equivalent vector equation (7) is $\mathcal{C}^k$ for all k beause it is a rational function without singularities. The Global Differentiable Dependence Theorem tells us that the solution $z(t, z, \mu)$ is differentiable wrt μ for all $(t, \bar{x}, \bar{\mu})$. The partial derivatives are the components

$$W(t) = D_\mu z(t, \bar{z}, \bar{\mu}) = \begin{pmatrix} \dfrac{\partial x}{\partial \mu}(t, \bar{z}, \bar{\mu}) \\ \dfrac{\partial y}{\partial \mu}(t, \bar{z}, \bar{\mu}) \end{pmatrix}.$$

To estimate the size of the partial derivatives, we may apply Gronwall's Inequality to the integrated version of the variational equation. For $t \ge 0$ such that $t \in I_{\bar{z},\bar{\mu}}$

$$\begin{aligned} |W(t)| &= \left| 0 + \int_0^t D_z f(z(s, \bar{z}, \bar{\mu}), \bar{\mu})\, W(s) + D_\mu f(z(s, \bar{z}, \bar{\mu})\, ds \right| \\ &\le \int_0^t |D_z f(z(s, \bar{z}, \bar{\mu}), \bar{\mu})\, W(s) + D_\mu f(z(s, \bar{z}, \bar{\mu})|\; ds \\ &\le \int_0^t \|D_z f(z(s, \bar{z}, \bar{\mu}), \bar{\mu})\|\, |W(s)| + |D_\mu f(z(s, \bar{z}, \bar{\mu})|\, ds \\ &\le \int_0^t \sqrt{\mu^2 + 3}\, |W(s)| + (|z_0| + t) e^{t\sqrt{\mu^2+2}}\, ds \\ &\le t(|z_0| + t) e^{t\sqrt{\mu^2+2}} + \int_0^t \sqrt{\mu^2 + 3}\, |W(s)|\, ds \end{aligned}$$

where we have used

$$D_z f(z(t,\bar z,\bar\mu),\bar\mu) = \begin{pmatrix} -\mu & 1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{2x}{(1+x^2)^2} & 0 \end{pmatrix}$$

$$D_\mu f(z(t,\bar z,\bar\mu) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} z(t,\bar z,\bar\mu)$$

that are estimated using $2|z| \le 1 + z^2$ and (10),

$$\|D_z f(z(t,\bar z,\bar\mu),\bar\mu)\| \le \sqrt{\mu^2+3},$$

$$|D_\mu f(z(t,\bar z,\bar\mu)| \le |z(t,\bar z,\bar\mu)| \le (|z_0| + t)e^{t\sqrt{\mu^2+2}}.$$

By the Gronwall Inequality, we find the desired estimates on the derivative. For $0 \le t$ such that $t \in I_{\bar z,\bar\mu}$,

$$|W(t)| \le t(|z_0| + t)e^{t\sqrt{\mu^2+2}} e^{t\sqrt{\mu^2+3}} \le t(|z_0| + t)e^{2t\sqrt{\mu^2+3}}.$$

4. *Consider the first order differential equation*

$$\frac{dx}{dt} = f(t,x),$$

where $f(t,x)$ is smooth and periodic in t: $f(t+1,x) = f(t,x)$ for all x and t in $\mathbf{R}$.

(a) *Define the Poincare map for this differential equation.*

(b) *Suppose $f(t,x) = A(t)x - x^2$, where $0 < \alpha < A(t) < \beta$. Prove that the differential equation has at least one nontrivial periodic solution.*

By the Global Existence theorem, the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \\ x(0) = x_0. \end{cases} \tag{14}$$

has a solution $x(t;x_0)$ which is defined whenever $t \in I_{x_0}$, the maximal interval of existence. The Poincaré Map in the context of $T = 1$ periodic solutions is the time-one map

$$\wp(x_0) = x(1;x_0)$$

provided $1 \in I_{x_0}$. There is a T-periodic solution $x(t,x_0)$ whenever x_0 is a fixed point of the Poincaré Map $x_0 = \wp(x_0)$.

In the special case $f(t,x) = A(t)x - x^2$ where $A(t)$ is a smooth ($\mathcal{C}^k$ for any k) $T = 1$ periodic real function. Under the hypothesis $0 < \alpha < A(t) < \beta$, we see that

$$f(\alpha) = A(t)\alpha - \alpha^2 > \alpha^2 - \alpha^2 = 0$$

$$f(\beta) = A(t)\beta - \beta^2 < \beta^2 - \beta^2 = 0.$$

We wish to show $\wp$ is defined for $x_0 \in [\alpha,\beta]$. First, we observe that for any $x_0 \in [\alpha,\beta]$, there is a $\tau > 0$ such that $\alpha < x(t,x_0) < \beta$ for $0 < t < \tau$. By the Global Existence Theorem (or Local Existence Theorem), we know that for all x_0, $x(t,x_0)$ is defined for t in a neighborhood of $t = 0$. By the continuity of solutions, if $\alpha < x_0 < \beta$ then there is a

$\tau > 0$ so that $\alpha < x(t, x_0) < \beta$ for $0 < t < \tau$. On the other hand if $x_0 = \beta$, we know that $\dot{x}(0, \beta) = f(0, \beta) < 0$. Since $x(t, \beta)$ is differentiable in t,

$$0 > f(0, \beta) = \frac{dx}{dt}(0, \beta) = \lim_{t \to 0} \frac{x(t, \beta) - \beta}{t - 0}.$$

Thus there is a $\tau > 0$ such that for every $0 < t < \tau$

$$\frac{x(t, \beta) - \beta}{t - 0} < 0$$

so $x(t, x_0) < \beta$. Similarly, if $x_0 = \alpha$ then there is a $\tau > 0$ such that $x(t, x_0) > \alpha$ whenever $0 < t < \tau$.

To finish the claim, we argue both that the solution exists for $t \in [0, 1]$ and that $\alpha \le x(t, x_0) \le \beta$ for all $t \in [0, 1]$. If for some $x_0 \in [\alpha, \beta]$ the maximal interval $I_{x_0} = (p, q)$ does not include $t = 1$ then $x(t, x_0)$ exits any compact set: there is $0 < t_1 < p < 1$ such that $|x(t_1, x_0)| > \beta$. If on the other hand for some $x_0 \in [\alpha, \beta]$ the trajectory exits the interval, then there is $0 < t_1 \le 1$ such that $x(t_1, x_0) \notin [\alpha, \beta]$. In either case $x(t_1, x_0) \notin [\alpha, \beta]$.

By continuity of $x(t, x_0)$ there is a first $0 < t_2 < t_1$ such that $x(t_2, x_0) \in \{\alpha, \beta\}$. In other words

$$t_2 = \sup \{\tau \in (0, t_1) : x(s, x_0) \in (\alpha, \beta) \text{ for all } 0 < s < \tau\}.$$

We have already shown that there is a $\tau > 0$ for every $x_0 \in [\alpha, \beta]$. By continuity $x(t_2, x_0) = \alpha$ or $x(t_2, x_0) = \beta$ and $x(s, x_0) \in (\alpha, \beta)$ for all $0 < s < \tau$.

Let us rule out the possibility $x(t_2, x_0) = \beta$. The argument in case $x(t_2, x_0) = \alpha$ is similar. Since $x(s, x_0) < \beta$ for $0 < s < t_2$ it follows that

$$\dot{x}(t_2, x_0) = \lim_{s \to t_2 -} \frac{x(s, x_0) - \beta}{s - t_2} \ge 0.$$

But this contradicts the ODE since

$$\dot{x}(t_2, x_0) = f(t_2, x(t_2, x_0)) = f(t_2, \beta) < 0.$$

This completes the claim that for $x_0 \in [\alpha, \beta]$ the solution exists for $0 \le t \le 1$ and that $x(t, x_0) \in [\alpha, \beta]$ for all $0 \le t \le 1$.

This implies that $\wp([\alpha, \beta]) \subset [\alpha, \beta]$. By the Global Differentiable Dependence Theorem (or Global Continuity Theorem) $\wp(x_0)$ is continuous in x_0. By the Intermediate Value Theorem, $\wp$ which maps a compact interval to itself has a fixed point $x_1 \in [\alpha, \beta]$ such that $x_1 = \wp(x_1)$. Thus the solution $x(t, x_1)$ is a nontrivial $T = 1$ periodic solution.

5. *Find a matrix T that such that $T^{-1}AT = J$, the Jordan form, and check your answer. Using your J, find the solution to $\dot{x} = Ax$, $x(0) = c$.*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \qquad c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

The eigenvalues are the diagonals $\lambda = 1$ with algebraic multiplicity three. An eigenvector satisfies

$$0 = (A - \lambda I)V_1 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

As $A - \lambda I$ has rank two, V_1 generates the one dimensional eigenspace. It follows that the Jordan form of this matrix is a 3×3 block

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = I + N.$$

To find T, we write the cyclic vectors by inspection.

$$(A - \lambda I)V_2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = V_1,$$

$$(A - \lambda I)V_3 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{3}{16} \\ \frac{1}{8} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} = V_2$$

Take

$$T = \big(V_1 \mid V_2 \mid V_3\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{16} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}.$$

To check $T^{-1}AT = J$ we compute

$$AT = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{16} \\ 0 & 0 & \frac{1}{8} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{5}{16} \\ 0 & 0 & \frac{1}{8} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{16} \\ 0 & 0 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = JT.$$

Using $IN = NI$,

$$e^{tJ} = e^{t(I+N)} = e^{tI}e^{tN} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = e^t \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

The solution of $\dot{x} = Ax$, $x(0) = c$ is given by

$$x(t) = e^{tA}c = e^{tTJT^{-1}}c = Te^{tJ}T^{-1}c = e^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{16} \\ 0 & 0 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= e^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{16} \\ 0 & 0 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 1 & 2t & 3t+4t^2 \\ 0 & 2 & 3+8t \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = e^t \begin{pmatrix} 1 & 2t & 3t+4t^2 \\ 0 & 1 & 4t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$