Homework for Math 6410 §1, Fall 2023

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Our main text this semester is Marcello Viana & Jose M. Espinar, *Differential Equations: A Dynamical Systems Approach to Theory and Practice,*Amer. Math. Soc., 2021, ISBN 9781470451141 (hardcover) ISBN 9781470465407 (paper back) ISBN 9781470465384 (ebook). Please read the relevant sections in the text as well as any cited reference. Each problem is due three class days after its assignment, or on Tuesday, Dec. 11, whichever comes first. Please write out the problem that you are answering on your solution papers.

1. [Aug. 21] **Compute a Phase Portrait using the Computer**

   This exercise asks you to figure out how to make a computer algebra system draw a phase portrait. For many of you this will already be familiar. See, *e.g.,* the MAPLE worksheet from today's lecture

   `http://www.math.utah.edu/~treiberg/M6412eg1.mws`

   `http://www.math.utah.edu/~treiberg/M6412eg1.pdf`

   or my lab notes from Math 2280,

   `http://www.math.utah.edu/~treiberg/M2282L4.mws`.

   Choose an autonomous system in the plane with at least two rest points such that one of the rest points is a saddle and another is a source or sink. (Everyone in class should have a different ODE.) Compute the Jacobian at the rest points. Find the eigenvalues of the Jacobians and determine the stability type at each rest point. Using your favorite computer computer algebra system, *e.g.,* MAPLE or MATLAB or any ODE solver, plot the phase portrait indicating the background vector field and enough integral curves to show the topological character of the flow. You should include the stable and unstable curves at the saddles, trajectories at the all rest points including any that connect the nodes, as well as any separatrices.

2. [Aug. 23] **Delay Differential Equation.**

   The delay differential equation involves past values of the unknown function x, and so its initial data φ must be given for times $t \le 0$. Using the Contraction Mapping Principle, show the local existence and uniqueness of a solution to the delay differential equation.

   Theorem. Let $\mathcal{U} \subset \mathbf{R} \times \mathbf{R}^d$ be an open set and $f : \mathcal{U} \to \mathbf{R}^d$ be a continuous function that satisfies a local Lipschitz condition in x. Let $\tau(t) : I \to [0.b]$ be a continuous function defined on the open interval $I \subset \mathbf{R}$. Let $(t_0, x_0) \in \mathcal{U}$. Let $\varphi : [t_0 - b, t_0] \to \mathbf{R}$ be a continuous function such that $\phi(t_0) = x_0$ and $(t, \phi(t)) \in \mathcal{U}$ for $t \in [t_0 - b, t_0]$. Then there is a number δ_0 and a unique continuous function $x : [t_0 - b, t_0 + \delta_0) \cap I \to \mathbf{R}$, differentiable on $[t_0, t_0 + \delta_0) \cap I$ satisfying the initial value problem

$$\begin{cases} \dfrac{dx}{dt}(t) = f\big(t, x(t - \tau(t))\big) & \text{for } t \in [t_0, t_0 + \delta_0) \cap I, \\ x(t) = \varphi(t) & \text{for all } t \in [t_0 - b, t_0]. \end{cases}$$

[*cf.* Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 71.]

3. [Sept. 25] **Alternative Proof of the Picard Theorem.**

   We set up the same as in the Picard Theorem. Let $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ be an open set and $(t_0, x_0) \in \Omega$. Suppose $f(t,x) \in \mathcal{C}(\Omega, \mathbf{R}^n)$ satisfies a local Lipschitz condition in x uniformly in t. Let $a, b > 0$ so that the closed cylinder $\Sigma = [t_0 - a, t_0 + a] \times \overline{B_b(x_0)} \subset \Omega$. Let

$$M = \sup_{(t,x)\in\Sigma} |f(t,x)|$$

   and L be the Lipschitz constant on Σ. Let $\epsilon = \min\{a, b/M\}$ and $I = [t_0 - \epsilon, t_0 + \epsilon]$. Let $X = \mathcal{C}(I, \mathbf{R}^n)$ be the Banach Space of continuous curves with sup-norm $\| \bullet \|$. Then $C = \{y \in X : |y(t) - x_0| \le b \text{ for all } t \in I\}$ is a closed subset. For $y \in C$ we defined the operator

$$T[y](t) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

   We have shown that $T : C \to C$. Put $y_0 = x_0$ and let $y_{n+1} = T(y_n)$ for all n. We have $y_n \in C$ for all n by induction.

   (a) Prove

$$|y_{n+1}(t) - y_n(t)| \le \frac{ML^n |t - t_0|^{n+1}}{(n+1)!} \qquad \text{for all } n = 0, 1, 2, \dots \text{ and } t \in I.$$

   (b) Using (a), show that there is a solution $x \in \mathcal{C}^1(I, \mathbf{R}^n)$ of the initial value problem

$$\dot{x}(t) = f\big(t, x(t)\big), \qquad \text{for } t \in I;$$
$$x(t_0) = x_0.$$

   Note that this improves our result using contraction mapping, exploiting the fact that the integral equation is of Volterra type. Your proof should not use Lemma 2.13 to prove Theorem 2.12 as done in Viana and Espinar's text.

4. [Aug. 28] **Existence via Schauder's Theorem.**

   Give another proof of the Peano Existence Theorem using the Schauder Fixed Point Theorem.

   Theorem. [Peano Existence Theorem] Let $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ be a domain and $f \in \mathcal{C}(\Omega, \mathbf{R}^n)$. Then for any $(t_0, x_0) \in \Omega$ there is $\epsilon > 0$ and a continuously differentiable function $x(t) : [t_0 - \epsilon, t_0 + \epsilon] \to \mathbf{R}^n$ which solves the initial value problem

$$\frac{dx}{dt} = f(t, (x(t))), \qquad \text{for all } t \in [t_0 - \epsilon, t_0 + \epsilon];$$
$$x(t_0) = x_0.$$

   Theorem. [Schauder Fixed Point Theorem] Let $\mathcal{A}$ be a closed, bounded, convex subset of a Banach space $\mathcal{X}$ and $T : \mathcal{A} \to \mathcal{A}$ be a completely continuous function. Then T has a fixed point in $\mathcal{A}$.

   A subset $\mathcal{K}$ of a Banach space is *compact* if any sequence in $\{\phi_i\}_{i=1,2,\dots} \subset \mathcal{K}$ has a subsequence that converges to an element in $\mathcal{K}$. f is *compact* if for any bounded set $\mathcal{B} \subset \mathcal{X}$, the closure of the set $f(\mathcal{B})$ is compact. f is *completely continuous* if it is both compact and continuous. [*cf.* Hale, p. 14.]

5. [Aug. 30] **Existence via Delay Differential Equations.**

   Give a third proof of the Peano Existence Theorem using the following ideas.

   Theorem. [Peano Existence Theorem] Let $f(t,x)$ be continuous on the rectangle $R = \{(t,x) : |t-t_0| \le b, |x-x_0| \le a\}$. Let $M = \sup_R |f(t,x)|$ and $\delta = \min\{b, a/M\}$. Then the initial value problem

$$\frac{dx}{dt} = f(t,(x(t))),$$
$$x(t_0) = x_0.$$

   has at least one solution defined for $|t-t_0| \le \delta$.

   Suggestion: Define (with a slight abuse of notation)

$$x_n(t) = \begin{cases} x_0, & \text{if } |t-t_0| \le \delta/n; \\ x_0 + \int_{t_0}^{t-\delta\,\mathrm{sgn}(t-t_0)/n} f(s, x_n(s))\,ds, & \text{if } \delta/n < |t-t_0| \le \delta. \end{cases}$$

   Show that x_n is well defined and apply the Arzela-Ascoli Theorem.

   [*cf.* Sideris, *Ordinary Differential Equations and Dynamical Systems,* Atlantis Press, 2013, p.48.]

6. [Sept. 1.] **Compare Solutions of Two Mathieu Equations.**

   One solution for Problem 19 of the 2010 Math 6410 depended on comparing the solutions of the perturbed and unperturbed problems. Find a sharp estimate for the difference in values and derivatives at $T = \frac{2\pi}{3}$ of the solutions for the two initial value problems, where u_0, u_1, ϵ are constants.

$$\begin{aligned} \ddot{x} + x &= 0, & \ddot{y} + (1+\epsilon\sin(3t))y &= 0, \\ x(0) &= u_0, & y(0) &= u_0, \\ \dot{x}(0) &= u_1; & \dot{y}(0) &= u_1. \end{aligned}$$

   Let $y(t; u_0, u_1, \epsilon)$ solve the IVP. Use your estimate to show $|y(T;1,0,\epsilon) + \dot{y}(T;0,1,\epsilon)| < 2$ for small $|\epsilon|$.

7. [Sept. 6.] **Generalize Gronwall's Inequality**

   Let I be an real interval and $t_0 \in I$. Let $\alpha(t)$, $\beta(t)$ and $u(t)$ be continuous nonnegative functions on I such that

$$u(t) \le \alpha(t) + \left| \int_{t_0}^t \beta(s)u(s)\,ds \right|$$

   for every $t \in I$. Show that

$$u(t) \le \alpha(t) + \left| \int_a^t \alpha(s)\beta(s)\exp\left(\left| \int_s^t \beta(\sigma)\,d\sigma \right| \right) ds \right|$$

   for every $t \in I$.

   [*cf.* Graef, Henderson, Kong and Liu, *Ordinary Differential Equations and Boundary Value Problems Vol. 1,* World Scientific, 2018, p. 13.]

8. [Sept. 8.] **Find a Periodic Solution.**

This exercise gives conditions for an ordinary differential equation to admit periodic solutions.

(a) Assume that $f \in C(\mathbb{R} \times [-1,1])$ such that for some $L < \infty$ and some $0 < T < \infty$ we have

$$\begin{gathered} |f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|, \\ f(T+t, y_1) = f(t, y_1), \\ f(t,-1)f(t,+1) < 0 \end{gathered}$$

for all $t \in \mathbb{R}$ and all $y_1, y_2 \in [-1,1]$. Show that the equation $y' = f(t,y)$ has at least one solution periodic of period T.

(b) What conditions does this require of $y' = a(t)y + b(t)$ to admit period solutions, where a, b are T periodic functions.

[*cf.* Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 101.]

9. [Sept. 11.] **Concrete Variational Equation.**

Let

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}.$$

Find the solution $\varphi(t,y) \in \mathbf{R}^3$ of

$$\begin{aligned} \frac{dx}{dt} &= f\big(x(t)\big), \\ x(0) &= y. \end{aligned}$$

Find

$$\Phi(t,y) = D_2\,\varphi(t,y).$$

Show that it satisfies the variational equation

$$\begin{aligned} \frac{d\Phi}{dt} &= Df\big(\varphi(t,y)\big) \cdot \Phi(t,y), \\ \Phi(0) &= I. \end{aligned}$$

[Perko, p. 84.]

10. [Sept. 13.] **Escape Times.**

Show that each solution $\big(x(t), y(t)\big)$ of the initial value problem

$$\begin{cases} x' = & x^2 + y \\ y' = & y^2 + x \end{cases} \qquad \begin{cases} x(0) = & x_0 \\ y(0) = & y_0 \end{cases}$$

with $x_0 > 0$ and $y_0 > 0$ cannot exist on an interval of the form $[0, \infty)$.

[*cf.* Wilson, *Ordinary Differential Equations,* Addison-Wesley, 1971, p.255.]

11. [Sept. 15.] **Global Continuity with Respect to Initial Point.**

    Let $\gamma : I \to \mathbf{R}^d$ be a maximal solution of the equation

$$\begin{aligned} \dot{x} &= f(t,x) \\ x(t_0) &= x_0 \end{aligned}$$

    Show that for every compact set $K \subset I$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(\bar{t}, \bar{x})$ in the ball of radius δ about (t_0, x_0), the domain of the maximal solution β with initial condition $\beta(\bar{t}) = \bar{x}$ contains K and $|\beta(t) - \gamma(t)| \le \varepsilon$ for all $t \in K$.

    [*cf.* Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 100.]

12. [Sept. 18.] **Global Existence.**

    Let Let $f : \mathbf{R}^{1+d} \to \mathbf{R}^d$ be a continuous function that satisfies $|f(t,x)| \le g(|x|)$ for some continuous and positive function $g : [0, \infty) \to \mathbf{R}$ satisfying

$$\int_0^\infty \frac{dr}{g(r)} = \infty.$$

    Prove that all maximal solutions satisfying $\dot{x} = f(t,x)$ with initial condition $x(t_0) = 0$ are defined for all $t \ge t_0$.

    [Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 102.]

13. [Sept. 20.] **Comparison Theorem.**

    Let $x(t)$ be a solution of the ODE $\dot{x} = f(t,x)$ where $f : [0,T) \times \mathbf{R} \to \mathbf{R}$ is $\mathcal{C}^!$ in (t,x). If a differentiable function $u(t)$ satisfies

$$\frac{du}{dt}(t) \le f(t, u(t)), \qquad \text{for all } 0 \le t < T,$$

    and the initial bound $u(0) \le x(0)$, then $u(t) \le x(t)$ for all $0 \le t < T$.

    [Schaeffer and Cain, Ordinary Differential Equations: Basics and Beyond, Springer 2016, p. 39.]

14. [Sept. 22.] **Autonomous Perturbation.** Let $f, g \in \mathcal{C}^1(\mathbb{R}^2)$. Consider the autonomous system

$$\begin{aligned} \dot{x} &= \frac{x}{\sqrt{x^2+y^2}} - x - y + \lambda f(x,y) \\ \dot{y} &= \frac{y}{\sqrt{x^2+y^2}} + x - y + \lambda g(x,y) \end{aligned}$$

    with special periodic solution $(\cos t, \sin t)$ for $\lambda = 0$. Prove that for small $|\lambda|$, periodic solutions exist close to $(\cos t, \sin t)$. [F. John, *Ordinary Differential Equations,* Courant Institute of Mathematical Science, 1965, p. 148.]

15. [Sept. 25.] **Accumulation Point of Periodic Orbits.**

    Let $F : \mathcal{U} \to \mathbf{R}^d$ be a $\mathcal{C}^1$ vector field and $p \in \mathcal{U}$ such that every neighborhood of p contains a periodic trajectory of F Conclude that p is a stationary point.

    [Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 188.]

16. [Sept. 27.] **Matrix Exponential**

    (a) Write the second order equation

    $$\ddot{u} + 2\dot{u} + u = 0 \tag{1}$$

    as a first order system, $\dot{x} = Ax$.

    (b) Find e^{tA} for the matrix of part (a).

    (c) Use the answer from part (b) to find the solution of (1) with $u(0) = u_0$ and $\dot{u}(0) = u_1$.

    [*cf.* Sideris, *Ordinary Differential Equations and Dynamical Systems,* Atlantis Press, 2013, p.18.]

17. [Sept. 29.] **Jordan Form.**

    Find the eigenvalues, generalized eigenvectors, cyclic vector chains and S such that $S^{-1}AS = J$, the Jordan form.

    $$A = \begin{pmatrix} 5 & 10 & 9 & 3 & 0 & 5 \\ -1 & -4 & -1 & -1 & 0 & 0 \\ 6 & 7 & 4 & 1 & 0 & 5 \\ -11 & -12 & -11 & -3 & 0 & -10 \\ 34 & 44 & 39 & 11 & -2 & 26 \\ -7 & -10 & -9 & -3 & 0 & -7 \end{pmatrix}$$

    [In Maple,

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    Matrix(6, 6, [[5, 10, 9, 3, 0, 5], [-1, -4, -1, -1, 0, 0], [6, 7, 4, 1, 0, 5],
    [-11, -12, -11, -3, 0, -10], [34, 44, 39, 11, -2, 26], [-7, -10, -9, -3, 0, -7]])
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18. [Oct. 2.] **Just Multiply by t.**

    Consider the n-th order constant coefficient linear homogeneous scalar equation

    $$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0$$

    where a_i are complex constants. Convert to a first order differential system $\mathbf{x}' = A\mathbf{x}$. Show that the geometric multiplicity of every eigenvalue of A is one. Show that a basis of solutions is $\{t^k \exp(\mu_i t)\}$ where $i = 1, \ldots, s$ correspond to the distinct eigenvalues μ_i and $0 \le k < m_i$ where m_i is the algebraic multiplicity of μ_i. [*cf.,* Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems,* Amer. Math. Soc., 2012, p.68.]

19. [Oct. 6.] **Operator Norm.**

    Verify that the operator norm $\|\bullet\|$ is a complete norm on the space $\mathcal{L}(\mathbf{R}^d, \mathbf{R}^d)$. Justify that if λ is an eigenvalue of B then $|\lambda| \le \|B\|$. Moreover, if B is invertible, then $|\lambda| \ge \|B^{-1}\|^{-1}$. Verify also that the map $B \mapsto \det(B)$ is continuous with respect to the operator norm.

    [Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 225.]

20. [Oct. 16.] **Avoiding the Jordan Form.**

    Sometimes the use of the Jordan Canonical Form and matrices with multiple eigenvalues can be avoided using the following considerations. Let $A \in \mathcal{L}(\mathbb{C}^n)$.

    (a) Show for any $\varepsilon > 0$ there is a matrix B with distinct eigenvalues so that $\|A - B\| < \varepsilon$.
    .

    (b) By a simpler algorithm than finding the Jordan Form, one can change basis by a P that transforms A to upper triangular, called the Schur Form of the matrix.

    $$P^{-1}AP = U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}. \tag{2}$$

    Show that this fact can be used instead of Jordan Form to characterize all solutions of $\dot{y} = Ay$ (as linear combinations of products of certain exponentials, polynomials and trigonometric functions). [*c.f.,* Bellman, *Stability Theory of Differential Equations*, pp. 21–25.])

    (c) Show that given $\epsilon > 0$ there exists a nonsingular P such that in addition to (2) we may arrange that $\sum_{i<j} |u_{ij}| < \epsilon$.

    (d) Give three different proofs that $\det(e^A) = e^{\operatorname{trace}(A)}$.

21. [Oct. 16.] **Contracting Linear Map.**

    Let $B : \mathbf{R}^d \to \mathbf{R}^d$ be a linear isomorphism such that all of its eigenvalues are less (respectively greater) than 1 in absolute value. Show that there is a norm on $\mathbf{R}^d$ with respect to which the operator norm of B (respectively B^{-1}) is less than one.

    [Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 227.]

22. [Oct. 18.] **A Growth Estimate for Second Order Equation.**

    Let $p, q : [a, b] \to \mathbf{R}$ be continuous fuctions such that $q(t) > 0$ for all $t \in [a, b]$, and let $\xi : [a, b] \to \mathbf{R}$ be the maximal solution of the IVP

    $$\ddot{x} = p(t)\dot{x} + q(t)x, \qquad x(a) = 0, \qquad \dot{x}(a) = 1.$$

    Suppose that $|p(t)| \le M$ for all $t \in [a, b]$. Show that $\xi(t) > 0$ for $t \in (a, b]$ and deduce that

    $$\xi(t) > \frac{1}{M}\left(1 - e^{-M(t-a)}\right) \qquad \text{for all } t \in (a, b].$$

    [Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 250.]

23. [Oct. 20.] **Variation of Parameters Formula.**

Solve the inhomogeneous linear system

$$\begin{cases} \dot{\mathbf{x}} = A(t)\,\mathbf{x} + \mathbf{b}(t), \\ \mathbf{x}(t_0) = \mathbf{c}; \end{cases}$$

where

$$A(t) = \begin{pmatrix} -2\cos^2 t & -1-\sin 2t \\ 1-\sin 2t & -2\sin^2 t \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hint: a fundamental matrix is given by

$$U(t,0) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 62.]

24. [Oct. 23.] **Application of Liouville's Theorem.**

Find a solution of the IVP for Bessel's Equation of order zero

$$\begin{cases} x'' + \dfrac{1}{t}\,x' + x = 0 \\ x(0) = 1, \qquad x'(0) = 0 \end{cases}$$

by assuming the solution has a power series representation (or use Frobenius Method.) Use Liouville's formula for the Wronskian to find a differential equation for a second linearly independent solution of the differential equation. Show that this solution blows up like $\log t$ as $t \to 0$. [*cf.* Fritz John, Ordinary Differential Equations, Courant Institute of Mathematical Sciences, 1965, p. 90.]

25. [Oct. 25.] ***T*-Periodic Linear Equations.**

Consider the T-periodic non-autonomous linear differential equation

$$\dot{x} = A(t)\,x, \qquad x \in \mathbf{R}^n, \qquad A(t+T) = A(t).$$

Let $X(t)$ be the fundamental matrix with $X(0) = I$.

(a) Show that there is at least one nontrivial solution $\chi(t)$ such that $\chi(t+T) = \rho\chi(t)$, where ρ is an eigenvalue of $X(T)$.

(b) Suppose that $X(T)$ has n distinct eigenvalues ρ_i, $i = 1, \ldots, n$. Show that there are n linearly independent solutions of the form $x_i = p_i(t)e^{\nu_i t}$ where $p_i(t)$ is T-periodic. How is ρ_i related to ν_i?

(c) Consider the equation $\dot{x} = f(t)\,A_0\,x$, $x \in \mathbf{R}^2$, with $f(t)$ a scalar T-periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet Multipliers.

[U. Utah PhD Preliminary Examination in Differential Equations, August 2008.]

26. [Oct. 27.] **Blowup in a Periodic Linear Equation.**

Let $\phi(t)$ be a real, continuous, π-periodic function. Consider the scalar equation

$$\ddot{x} - (\cos^2 t)\dot{x} + \phi(t)\, x = 0.$$

Show that there is a real solution that tends to infinity as $t \to \infty$.

[*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations,* Prentice Hall 2003, p. 162.]

27. [Oct. 30] **Stability of Linear Flows.**

Let A and B be 2×2 real matrices.

(a) Prove the following statements about solutions of $\dot{z} = Az$.

i. If $\operatorname{tr} A < 0$ and $\det A > 0$ then the origin is asymptotically stable.

ii. If $\operatorname{tr} A < 0$ and $\det A = 0$ then the origin is Liapunov stable.

iii. If $\operatorname{tr} A = 0$ and $\det A > 0$ then the origin is Liapunov stable.

iv. In all other cases the origin is unstable.

(b) If the origin is asymptotically stable for $\dot{z} = Az$, prove that there exists $\delta > 0$ (depending on A) such that if $\|A - B\| < \delta$ then the origin is asymptotically stable for $\dot{z} = Bz$.

(c) Is the preceding statement true if "asymptoticlly stable" is replaced by "Liapunov stable"?

[*cf.* Sideris, *Ordinary Differential Equations and Dynamical Systems,* Atlantis Press, 2013, pp. 50–51.]

28. [Nov. 1.] **Linearized Stability of Fixed Points.**

The SIR model of epidemics of Brauer and Castillo-Chávez relates three populations, $S(t)$ the susceptible population, $I(t)$ the infected population and $R(t)$ the recovered population. The other variables are positive constants. Assume that births in the susceptible group occur at a constant rate μK. Assume that there is a death rate of $-\mu$ for each population. Assume also that there is an infection rate of people in the susceptible population who become infected which is proportional to the contacts between the two groups βSI. There is a recovery of γI from the infected group into the recovered group. Finally, the disease is fatal to some in the infected group, which results in the removal rate $-\alpha I$ from the infected population. The resulting system of ODE's is

$$\begin{aligned}\dot{S} &= \mu K - \beta SI - \mu S\\ \dot{I} &= \beta SI - \gamma I - \mu I - \alpha I\\ \dot{R} &= \gamma I - \mu R\end{aligned}$$

(a) Note that the first two equations decouple and can be treated as a 2×2 system. Then the third equation can be solved knowing $I(t)$. Let $\delta = \alpha + \gamma + \mu$. For the 2×2 system, find the nullclines and the fixed points.

(b) Check the stability of the nonnegative fixed points. Show that for $\beta K < \delta$ the disease dies out. Sketch the nullclines and some trajectories in the phase plane in this case.

(c) Show that for $\beta K > \delta$ the epidemic reaches a steady state. Sketch the nullclines and some trajectories in the phase plane now.

[From R. C. Robinson, *An Introduction to Dynamical Systems*, Pearson 2004.]

29. [Nov. 3.] **Condition for Asymptotic Stability.**

Suppose that the zero solution to $\dot{x} = Ax$ is asymptotically stable. Let $g(t,x) \in \mathcal{C}^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ satisfy $g(t,0) = 0$ and

$$|g(t,x)| \le h(t)|x|, \qquad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}^n,$$

where $h(t)$ satisfies for positive constants k and r,

$$\int_0^t h(s)\,ds \le kt + r, \qquad \text{for all } t \ge 0.$$

Show that there is a constant $k_0(A) > 0$ such that if $k \le k_0$, then the zero solution of

$$\dot{x} = Ax + g(t,x)$$

is asymptotically stable. [*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations,* Prentice Hall 2003, p. 243.]

30. [Nov. 6.] **Liapunov Functions.**

Use a Liapunov Function to show that the zero solution is asymptotically stable

$$\ddot{x} + (2 + 3x^2)\,\dot{x} + x = 0.$$

Hint: A sneaky way is to show that this equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y - x^3 \\ \dot{y} &= -x + 2x^3 - 2y. \end{aligned}$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 348–349.]

31. [Nov. 8.] **La Salle's Invariance Principle.**

Let $b : \mathbf{R} \to \mathbf{R}$ and $c : \mathbf{R} \to \mathbf{R}$ be continuous functions such that $yb(y) > 0$ for all $y \ne 0$ and $xc(x) > 0$ for all $x \ne 0$ (this implies $b(0) = c(0) = 0$). Using the function

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x c(s)\,ds,$$

show that the zero solution of

$$\ddot{x} + b(\dot{x}) + c(x) = 0$$

is asymptotically stable. Moreover, it is globally asymptotically stable if we assume that

$$\int_0^x c(s)\,ds \to +\infty \qquad \text{as } x \to \pm\infty.$$

Find functions b and c for which the linearization stability method is inconclusive.

[Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 306.]

32. [Nov. 10.] **Hartman-Grobman Theorem.**

Find a homeomorphism h in a neighborhood of 0 that establishes an topological conjugacy between the flow of the differential system and the flow of the linearized system, *i.e.,* $h(\psi(t,x)) = e^{tA}h(x)$ where $A = Df(0)$ and $\psi(t,x_0)$ is the solution of $\dot{\mathbf{x}} = f(\mathbf{x})$, the nonlinear system given by

$$\begin{aligned}\dot{x} &= -x\\ \dot{y} &= -y + xz,\\ \dot{z} &= z.\end{aligned}$$

[In §4.2, Barreira & Valls discuss a proof, but you can guess h from the solutions and verify.]

33. [Nov. 13.] **Shooting Method.**

Theorem. *Suppose $f : [a,b] \times \mathbf{R}^2 \to \mathbf{R}$ is $\mathcal{C}^1$ and there exists an $M > 0$ such that*

$$\frac{\partial f}{\partial x}(t,x,y) > 0 \quad \text{and} \quad \left|\frac{\partial f}{\partial y}(t,x,y)\right| \le M \quad \text{for all } (t,x,y) \in [a,b] \times \mathbf{R}^2.$$

Then the boundary value problem

$$\begin{aligned}\ddot{x} &= f(t,x,\dot{x})\\ x(a) = x_a,& \quad x(b) = x_b\end{aligned}$$

admits a unique solution.

Use the following itinerary to prove this theorem.

(a) For each $v \in \mathbf{R}$, let $x_v : [a,b] \to \mathbf{R}$ be the solution to the IVP $\ddot{x} = f(t,x,\dot{x})$ with initial conditions $x(a) = x_a$ and $\dot{x}(a) = v$. Justify that $\xi_v(t) = \dfrac{\partial x_v}{\partial v}(t)$ satisfies a differential equation of the form $\ddot{\xi} = p(t)\dot{\xi} + q(t)\xi$ with $q(t) > 0$ for all $t \in [a,b]$.

(b) Use exercize 22 to show that the value of $\xi(b)$ is positive and uniformly bounded away from zero, and conclude that for each x_b there exists exactly one value of v such that $x_v(b) = x_b$.

[Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 345.]

34. [Nov. 15.] **Stable and Unstable Manifolds.**

Find the stable manifold W^s and unstable manifold W^u near the origin of the system

$$\begin{aligned}\dot{x} &= -x\\ \dot{y} &= -y + x^2\\ \dot{z} &= z + y^2.\end{aligned}$$

[*cf.* Perko, *Differential Equations and Dynamical Systems,* Springer, 1991, p. 116–117.]

35. [Nov. 17] **Series to Approximate Unstable Manifold**

$P = (-1, 0)$ is the only fixed point of the system

$$\dot{x} = x + e^{-y}$$
$$\dot{y} = -y$$

Sketch the phase portrait. Find the stable and unstable manifolds at P. To find the stable manifold, introduce a new variable $u = x + 1$ and write the stable manifold as

$$y = a_1 u + a_2 u^2 + a_3 u^3 + \mathbf{O}(u^4)$$

Check that your manifolds are tangent to the stable and unstable spaces at P.

[Strogatz, *Nonlinear Dynamics and Chaos,* Perseus Publ., 1994, p. 181.]

36. [Nov. 20.] **Center Manifold.**

Find a center manifold for the system

$$\dot{x} = -xy$$
$$\dot{y} = -y + x^2 - 2y^2$$

through the rest point at the origin. Find a differential equation for the dynamics on the center manifold. Show that every nearby solution is attracted to the center manifold. Determine the stability of the origin.

Hint: Look for a center manifold that is a graph $y = \psi(x)$ of the form

$$\psi(x) = \sum_{k=2}^{\infty} a_k \, x^k$$

using the condition of invariance $\dot{y} = \psi'(x)\dot{x}$ and $\psi(0) = \psi'(0) = 0$. Find the first few terms of the expansion, guess the rest and check. Then get the equation for the induced flow on the center manifold. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 304.]

37. [Nov. 22.] **Periodic Solution in the Brusselator System.**

Show that there is a nonconstant periodic trajectory for the system

$$\dot{x} = 1 - 4x + x^2 y$$
$$\dot{y} = 3x - x^2 y$$

[University of Utah Ph.D. Preliminary Examination in Differential Equations, August 2004.]

38. [Nov. 27.] **Periodic Orbit Attracts its Neighbors.**

Let $\mathcal{U} \subset \mathbf{R}^2$ be an open set and γ be an isolated periodic orbit of a $\mathcal{C}^1$ vector field $F : \mathcal{U} \to \mathbf{R}^2$. Prove that there exist a neighborhood $\mathcal{V}$ of γ such that for all $p \in \mathcal{V}$, either $\alpha(p) = \gamma$ or $\omega(p) = \gamma$.

[Viana and Espinar, Differential Equations: A Dynamical Systems Approach to Theory and Practice, Amer. Math. Soc. 2020, p. 425.]

39. [Nov. 29.] **Dulac's Criterion.**

Prove the following theorem.
Theorem. Let $A \subset \mathbf{R}^2$ be an annular domain. Let $f \in \mathcal{C}^1(A, \mathbf{R}^2)$ and let $\rho \in \mathcal{C}^1(A, \mathbb{R})$. Show that if $\operatorname{div}(\rho f)$ is both not identically zero and does not change signs in any open subset of A then the equation $x' = f(x)$ has at most one periodic solution in A.

Use this to show that the van der Pol oscillator ($\lambda = \text{const.} \neq 0$)

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + \lambda(1 - x^2)y\end{aligned}$$

has at most one limit cycle in the plane. Hint: let $\rho = (x^2 + y^2 - 1)^{-1/2}$.

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 90.]

40. [Dec. 1.] **Bifurcation in System.**

Discuss the bifurcations of the equilibrium points of

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - 2x^2 - \mu x^3.\end{aligned}$$

for $-\infty < \mu < \infty$. Sketch the bifurcation diagram in the (μ, x) plane. Cobnfirm that there is a fold bifurcation at $\mu = 1$. What happens at $\mu = 0$?

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 436.]

41. [Dec. 4.] **Extended Center Manifold.**

Find the value of μ for which there is a bifurcation at the origin for the system

$$\begin{aligned}\dot{x} &= y - x - x^2 \\ \dot{y} &= \mu x - y - y^2.\end{aligned}$$

(a) Find the evolution equation on the extended center manifold correct to third order.
(b) What is the nature of the bifurcation?

[Glendenning, *Stability, Instability and Chaos,* Cambridge U. Press, 1994, p. 246.]