Math 5470 § 1.	**©R Example:**	Name: Example
Treibergs	**Dynamics of Maps and Chaos**	Jan. 12, 2016

This is discussed in the chapter on "One Dimensional Maps," [Strogatz 1994, pp. 348–365]. To demonstrate the idea of a dynamical system, we consider the dynamics given by a continuously differentiable map

$$f : I \to I$$

where $I = [0, 1]$ is the closed unit interval. For us today, we consider the *logistic* map

$$f(x) = rx(1 - x)$$

where r is a real parameter. To make sure $f(I) \subset I$ we require that $0 \le r \le 4$. the map defines the motions on I. Starting from an initial point $x_1 \in I$ we define recursively

$$x_{k+1} = f(x_k)$$

The sequence $\{x_1, x_2, x_3, \ldots\}$ is called the *orbit* of the point x_1.

If $r \le 1$ then $f(x) < x$ for $0 < x \le 1$ so that $x_{n+1} = f(x_n) < x_n$ (unless all $x_i = 0$.) Hence x_n is a decreasing sequence $x_n \to 0$ as $n \to \infty$. The convergence is to the unique fixed point of the map, the point $x^* \in I$ such that

$$x^* = f(x^*).$$

If $r > 1$ then there are more fixed points. Indeed, solving $x^* = f(x^*)$ yields

$$0 = x^*(r - 1 - rx^*)$$

or

$$x^* = 0 \qquad \text{or} \qquad x^* = \frac{r-1}{r}$$

which is in I since $r > 1$.

Note that x^* is an attractive fixed point for $1 < r < 3$. We show that if x_k is close to x^* then x_{k+1} is even closer. Writing $x_k = x^* + \eta_k$ we see using Taylor's formula that

$$x^* + \eta_{k+1} = x_{x+1} = f(x_k) = f(x^* + \eta_k) = f(x^*) + f'(x^*)\eta_k + \mathbf{O}(\eta_k^2)$$

as $\eta_k \to 0$. As long as $|f'(x^*)| < 1$ then $|\eta_{k+1}| < |\eta_k|$ (for $|\eta_k|$ small enough) so x_{k+1} is closer to x^* than x_k so x^* is an *attractive* fixed point.

For the logistic map, $f' = r - 2rx$ so that

$$f'(x^*) = r - 2r\left(\frac{r-1}{r}\right) = 2 - r$$

so that $|f'(x^*)| < 1$ for $1 < r < 3$. If $r > 3$ then x^* is unstable and the orbit will not limit to x^* unless $x_1 = x^*$. Instead the map $f(f(x))$ has two new stable fixed points $p, q \in I$ and so these become limit points of orbits of period two. The dynamical system undergoes a period doubling bifurcation at $r = 3$.

$$f(f(x)) = rf(x)\big(1 - f(x)\big) = r^2x(1 - x)\big(1 - rx + rx^2\big)$$

The fixed points of $f \circ f$ are roots of $x = f(f(x))$. Factoring, we find

$$0 = x - f(f(x)) = x - r^2x(1 - x)\big(1 - rx + rx^2\big) = x(1 - r + rx)(1 + r - rx - r^2x + r^2x^2)$$

so

$$x^* = 0 \qquad \text{or} \qquad x^* = \frac{r-1}{r} \qquad \text{or} \qquad p, q = \frac{1 + r \pm \sqrt{(r-3)(r+1)}}{2r}$$

This example is done in ©**R**. First we plot $y = x$, $y = f(x)$ for $r = 2.9$ and $y = x$, $y = f(f(x))$ for $r = 3.2$.

Second we compute the sequence x_n for the staring values $x_1 = 1/\pi$ and for various r. The zig-zag line from $(1, x_1)$ to $(2, x_2)$ to $(3, x_3)$ and so on shows the dynamics. We consider several cases: when there is an attractive fixed point, after the first few period doubling bifurcations, and in the chaos region $r > r_\infty = 3.569946\ldots$ For $r = 3.83$, the orbit has period three. For $r = 3.828$ near there, there is intermittent behavior: the orbit has period three or so for a long time and then there is a spurt of chaotic activity until it settles down again.

R Session:

```
R version 3.2.1 (2015-06-18) -- "World-Famous Astronaut"
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Platform: x86_64-apple-darwin14.5.0 (64-bit)

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'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

[Workspace loaded from /home/1004/ma/treibergs/.RData]

> # Plot y=f(x)
> x=0:299/299
> r=2.9; f = function(x){r*x*(1-x)}
> plot(x,f(x),ylim=0:1,type="l"); abline(0,1,lty=4)
>
># Plot y=f(f(x))
> r=3.2; f = function(x){r*x*(1-x)}
> plot(x,f(f(x)),ylim=0:1,type="l"); abline(0,1,lty=4)
>
> # Iterate the map. print the first 21  c[n]'s
> r=2.9; c[1]=1/pi; for(k in 1:20){c[k+1]=r*c[k]*(1-c[k])};c;
 [1] 0.3183099 0.6292672 0.6765409 0.6346166 0.6724473 0.6387596 0.6691628 0.6420135
 [9] 0.6665133 0.6445926 0.6643696 0.6466496 0.6626323 0.6482972 0.6612231 0.6496207
[17] 0.6600796 0.6506861 0.6591517 0.6515451 0.6583988
```
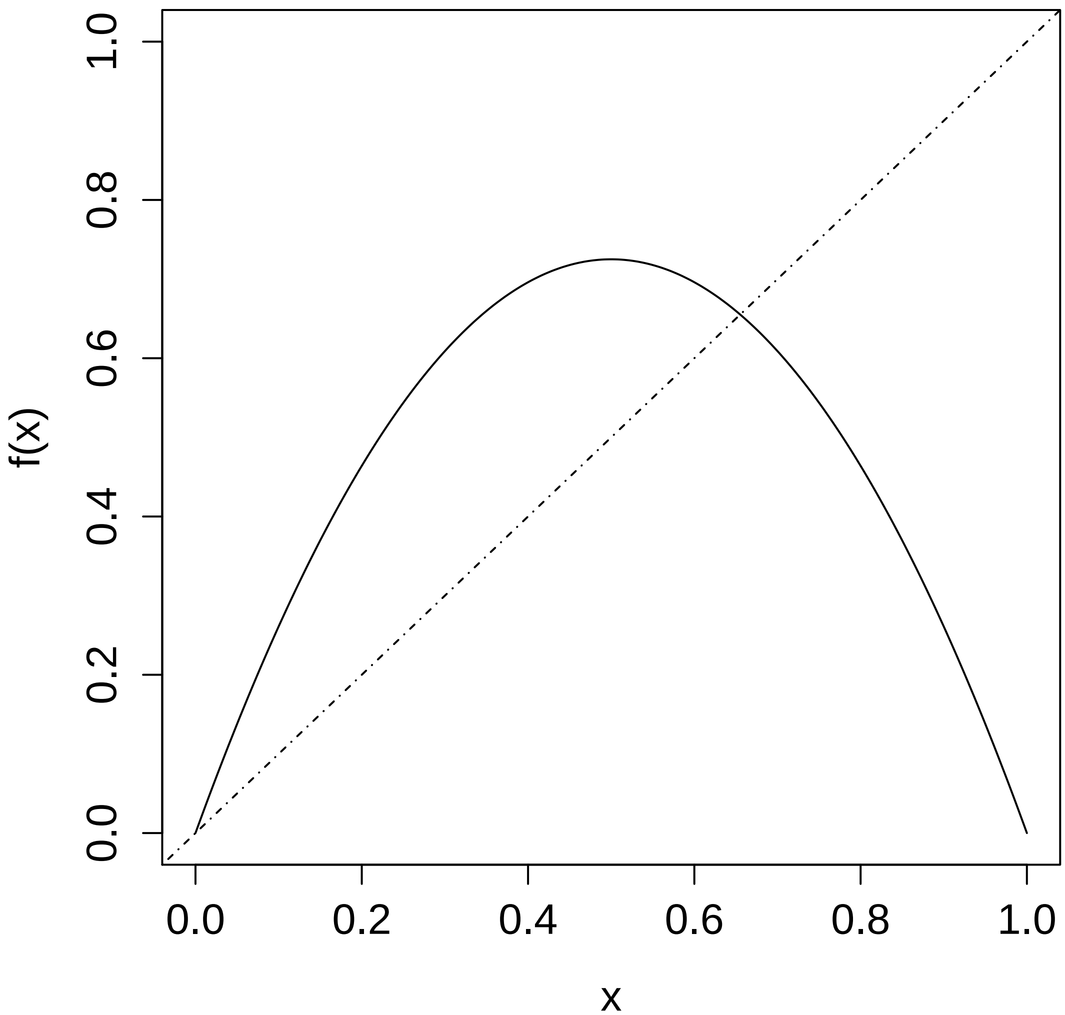
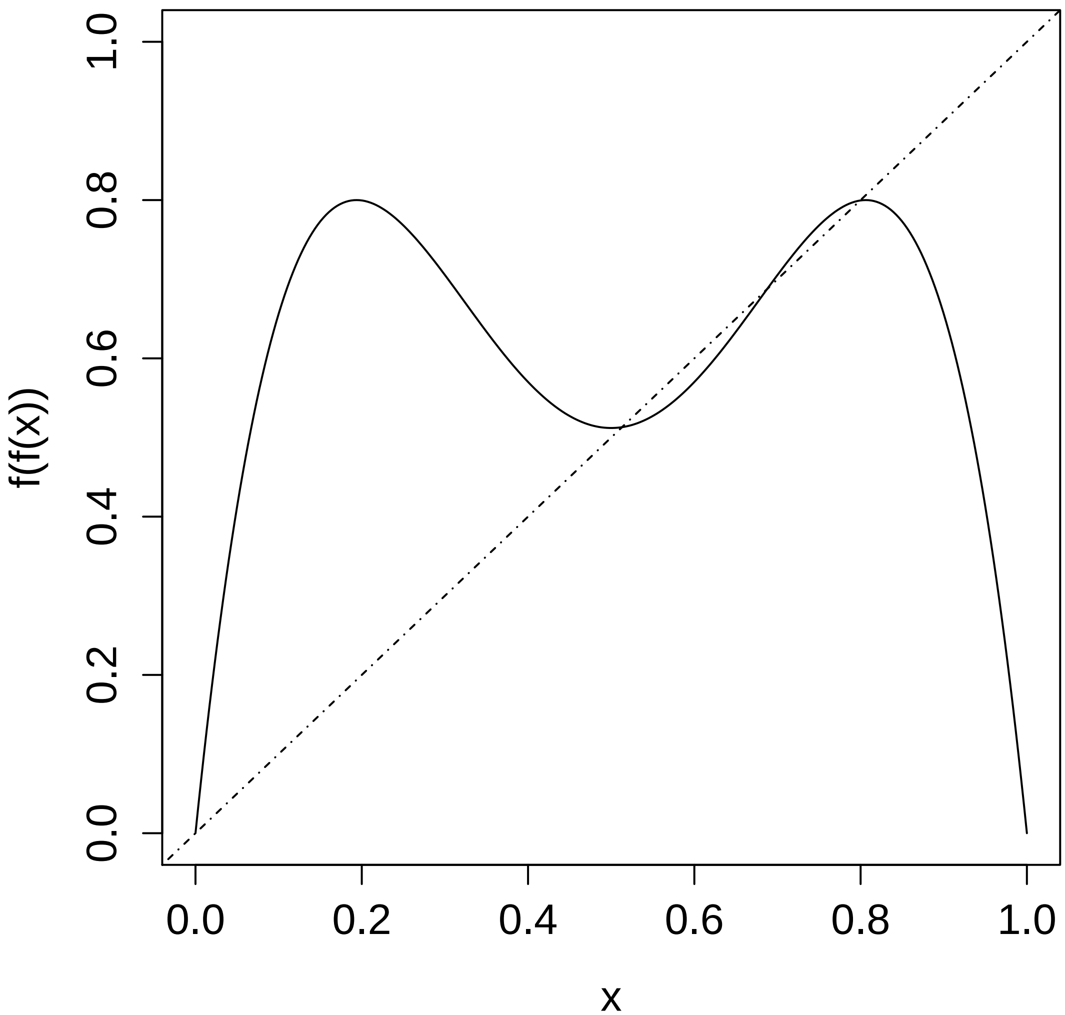
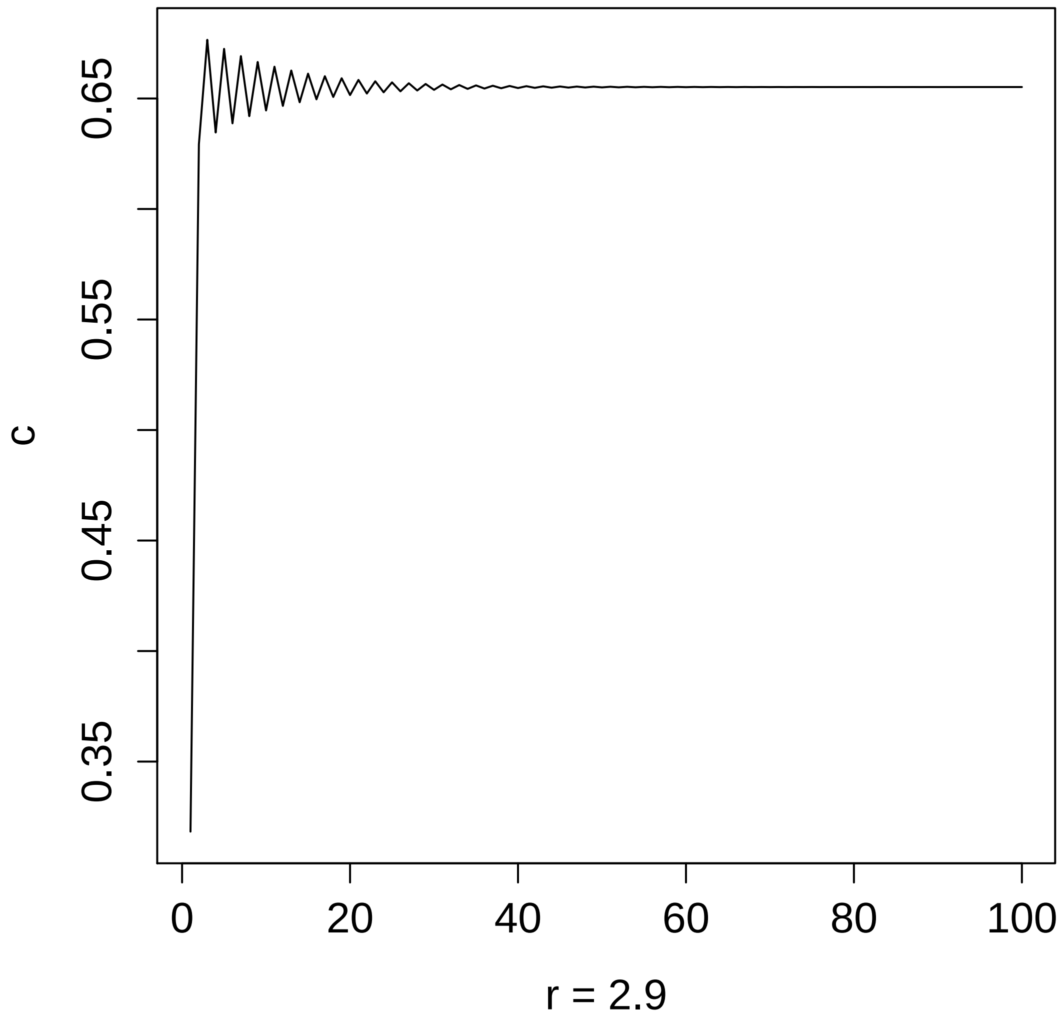
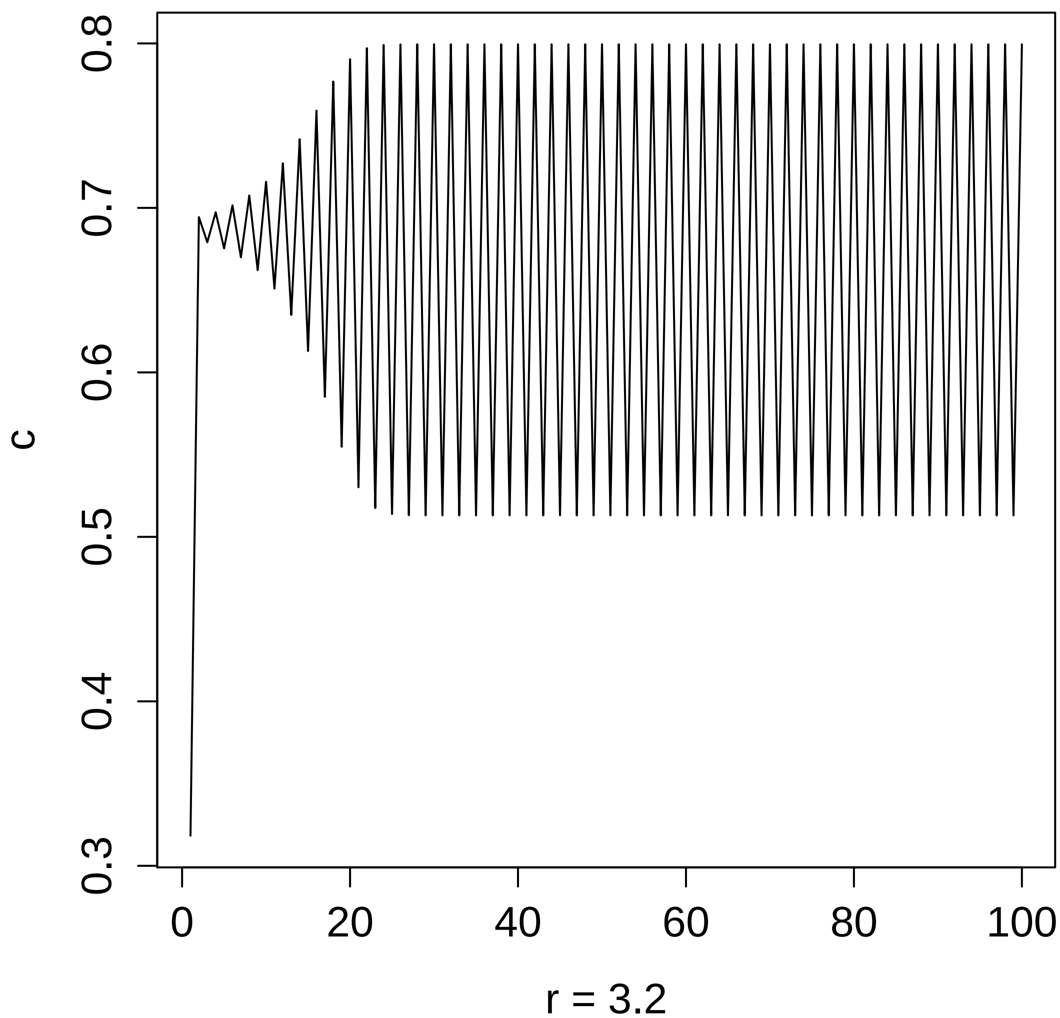
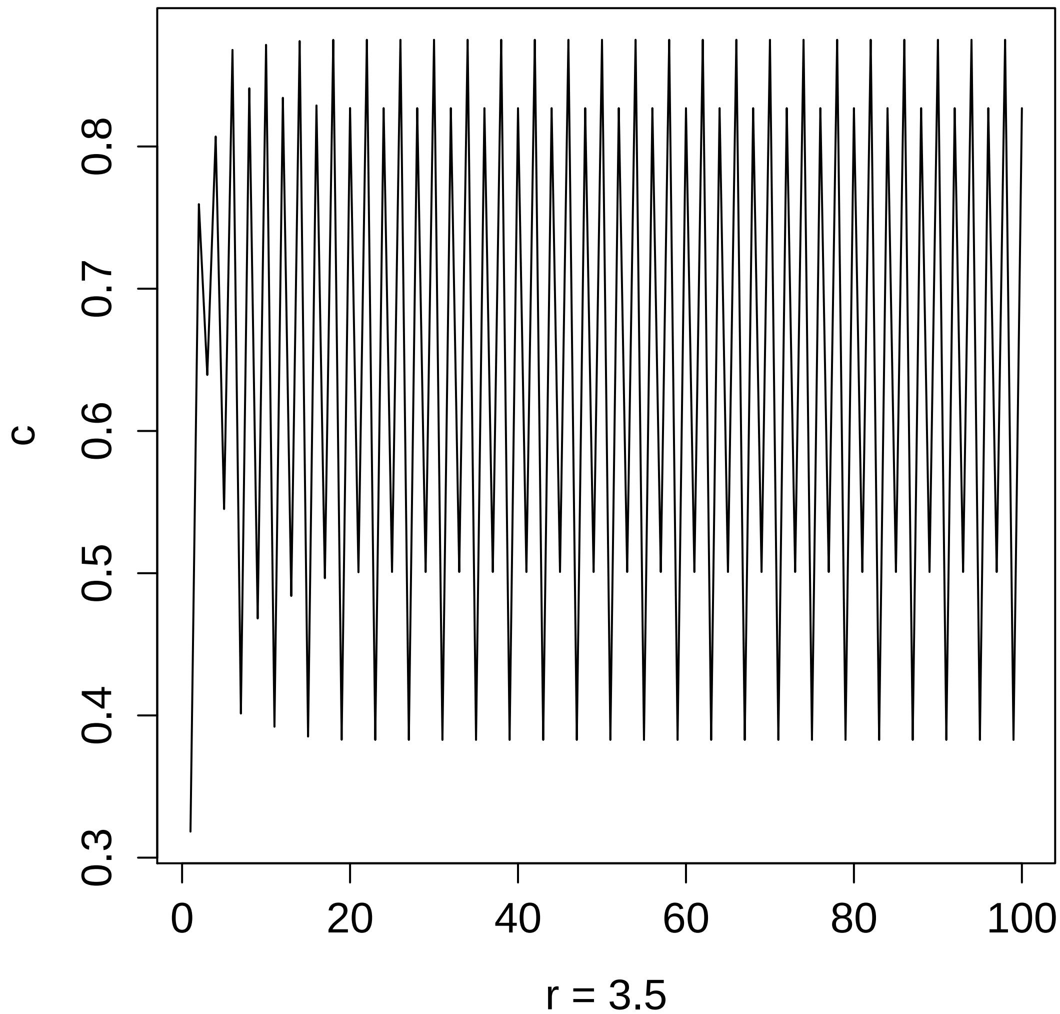

```
> # Iterate with different  r  values. Plot points  (n,c[n])  connected by lines.
>
> r=2.9; c[1]=1/pi; for(k in 1:99){c[k+1]=r*c[k]*(1-c[k])};plot(c,type="l",xlab="r = 2.9")
>
> r=3.2; c[1]=1/pi; for(k in 1:99){c[k+1]=r*c[k]*(1-c[k])};plot(c,type="l",xlab="r = 3.2")
>
> r=3.5; c[1]=1/pi; for(k in 1:99){c[k+1]=r*c[k]*(1-c[k])};plot(c,type="l",xlab="r = 3.5")
>
> r=3.55; c[1]=1/pi; for(k in 1:99){c[k+1]=r*c[k]*(1-c[k])};plot(c,type="l",xlab="r=3.55")
> r=3.83; c[1]=1/pi; for(k in 1:99){c[k+1]=r*c[k]*(1-c[k])};plot(c,type="l",xlab="r=3.83")
>
> r=3.828; c[1]=1/pi; for(k in 1:99){c[k+1]=r*c[k]*(1-c[k])};plot(c,type="l",xlab="r = 3.828")
```



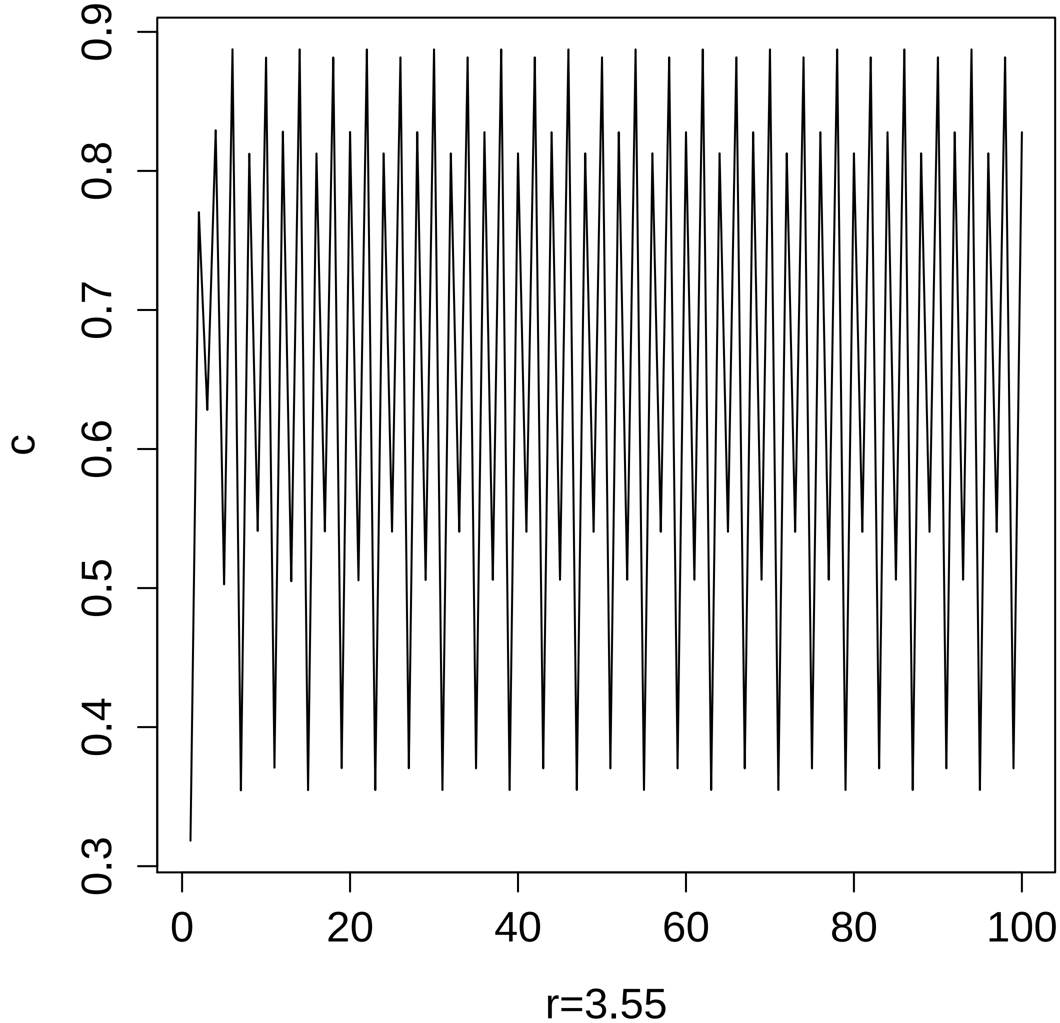

1.0
0.8
0.6
0.4
0.2
0.0
f(f(x))
0.0
0.2
0.4
0.6
0.8
1.0
x

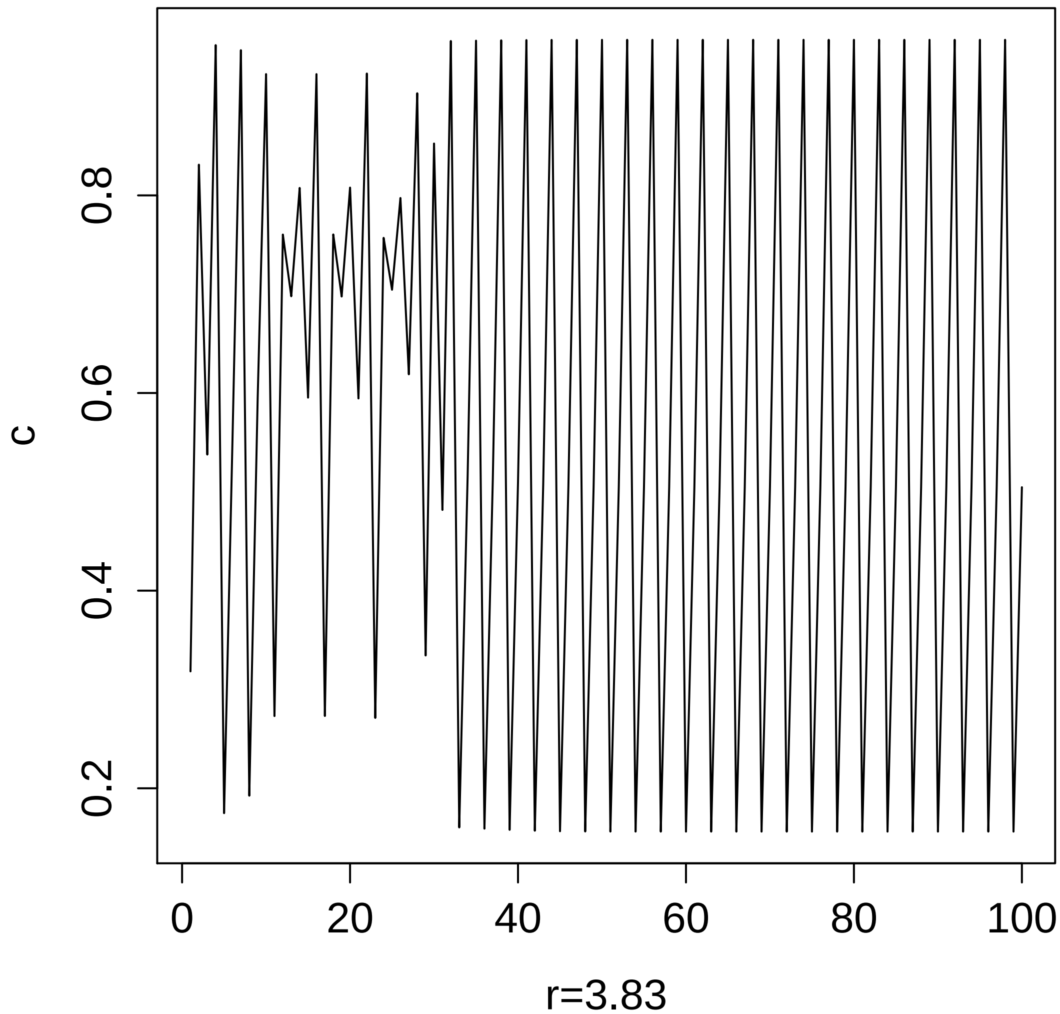

0.35
0.45
0.55
0.65
c
0
20
40
60
80
100
r = 2.9

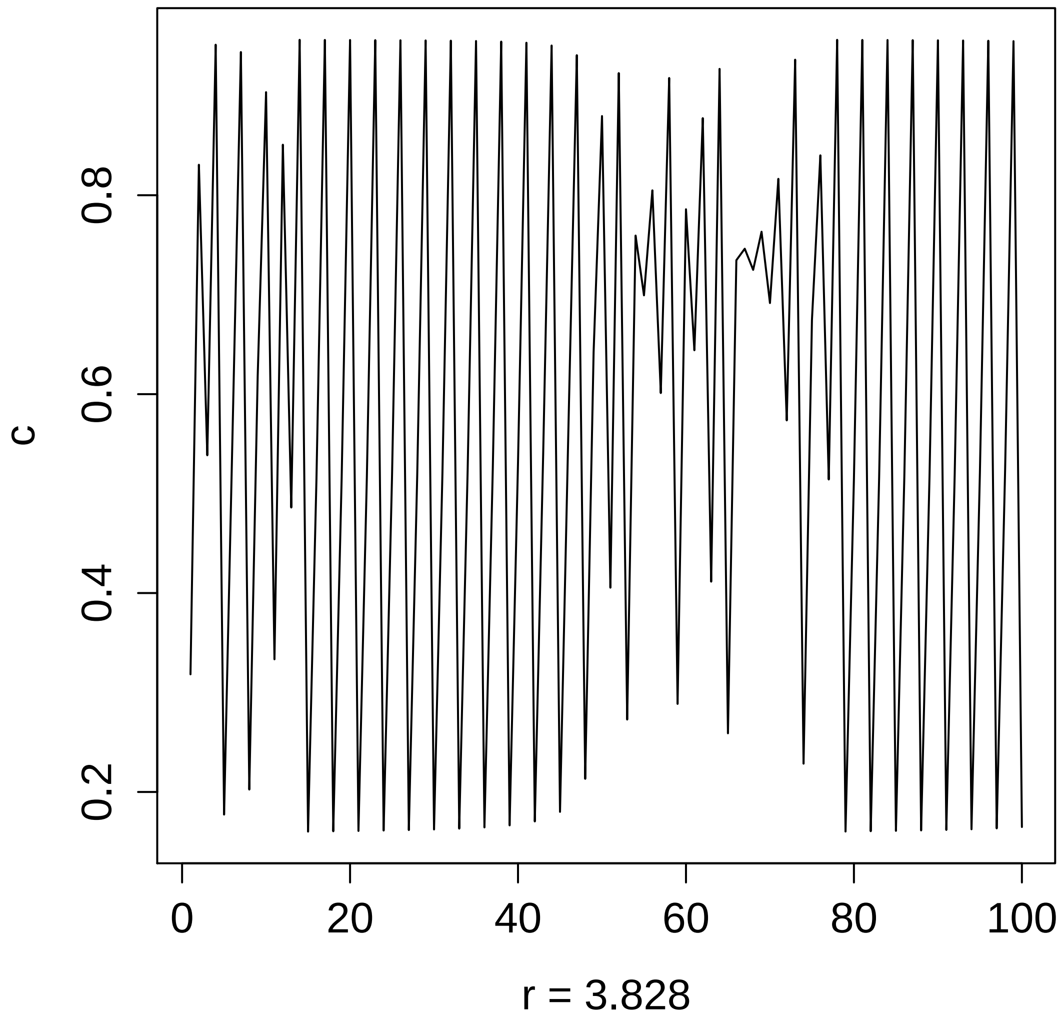

c
0.3
0.4
0.5
0.6
0.7
0.8
0
20
40
60
80
100
r = 3.2



c
0.3
0.4
0.5
0.6
0.7
0.8
0
20
40
60
80
100
r = 3.5


0.3
0.4
0.5
0.6
0.7
0.8
0.9
c
0
20
40
60
80
100
r=3.55


c
0.2
0.4
0.6
0.8
0
20
40
60
80
100
r=3.83


c
0.2
0.4
0.6
0.8
0
20
40
60
80
100
r = 3.828