

Homework for Math 5410 §1, Fall 2022

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November 17, 2022

Our text is by Morris Hirsch, Stephen Smale & Robert Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos* 3rd ed., Academic Press, 2013. Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on Dec. 3, whichever comes first.

Homework is to be written on paper. Your written work reflects your professionalism. Make answers complete and self contained. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Homework from Wednesday to Tuesday will be due Friday. Homework is to be handed in by 4:00 pm, Fridays to be considered on time. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all.

Please hand in problems A on Friday, August 26.

A. Exercises from the text by Hirsch, Smale & Devaney:

16[1-4, 9, 14]

Please hand in problems B on Friday, Sept. 2.

B. Exercises from the text by Hirsch, Smale & Devaney:

37[2d, 7, 9, 10, 11, 14]

Please hand in problems C on Friday, Sept. 9.

C. Exercises from the text by Hirsch, Smale & Devaney:

57[4, 5, 7, 10, 14]

71[1, 2, 3, 4, 5]

Please hand in problems D on Friday, Sept. 16.

D. Exercises from the text by Hirsch, Smale & Devaney:

103[4, 5 any two, 6, 7, 11]

Please hand in problems E on Friday, Sept. 23.

E. Exercises from the text by Hirsch, Smale & Devaney:

103[13, 14, 15]
135[1 any three]

Please hand in problems F1 – F2 on Friday, Sept. 30.

F1. Exercises from the text by Hirsch, Smale & Devaney:

135[4, 9, 12aej, 13]

F2. Solve the initial value problem:

$$\frac{d}{dt}X = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}X + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}; \quad X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Please hand in problems G1 – G3 on Friday, Oct. 7.

Reminder: your project outlines are due Oct. 21.

G1. Exercises from the text by Hirsch, Smale & Devaney:

157[1a, 2, 5]
407[5]

G2. Show that the iteration scheme $\psi_0(t) = A$,

$$\psi_{n+1}(t) = A + Bt + \int_0^t (s-t)\psi_n(s) ds$$

will converge to a solution of the problem $\ddot{x} + x = 0$, $x(0) = A$, $\dot{x}(0) = B$ for certain values of t . For what values of t is convergence assured? [From H. K. Wilson, Ordinary Differential Equations, Addison-Wesley, 1971, p.245.]

G3. The Contraction Mapping Principle. Here is the abstract idea behind the Picard Theorem. Let $V \subset \mathbf{R}^n$ be a closed subset. Let $0 < b < \infty$ and $0 < k < 1$ be constants and let $T : V \rightarrow V$ be a transformation. Suppose that for any $\phi, \psi \in V$ if $|\psi| \leq b$ then $|T(\psi)| \leq b$ and if both $|\phi| \leq b$ and $|\psi| \leq b$ then

$$|T(\psi) - T(\phi)| \leq K|\psi - \phi|,$$

i.e., T is a *contraction*. Prove that there exists an element $\eta \in V$ with $|\eta| \leq b$ such that $\eta = T(\eta)$, that is, T has a fixed point. Prove that η is the unique fixed point among points in V satisfying $|\eta| \leq b$. [Coddington & Levinson, Theory of Ordinary Differential Equations, Krieger 1984, pp. 40-41.]

Please hand in problems H1 – H4 on Friday, Oct. 22.

H1. Exercises from the text by Hirsch, Smale & Devaney:

184[1, 2]

H2. Let

$$f \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}.$$

Find the solution $\varphi(t, y) \in \mathbf{R}^3$ of

$$\begin{aligned} \frac{dx}{dt} &= f(x(t)), \\ x(0) &= y. \end{aligned}$$

Find

$$\Phi(t, y) = D_2 \varphi(t, y).$$

Show that it satisfies the variational equation

$$\begin{aligned} \frac{d\Phi}{dt} &= Df(\varphi(t, y)) \cdot \Phi(t, y), \\ \Phi(0) &= I. \end{aligned}$$

[Perko, p. 84.]

H3. Let a, b and p be positive constants. Consider the differential equation

$$\begin{aligned} \dot{x} &= -\frac{ax}{\sqrt{x^2 + y^2}} \\ \dot{y} &= -\frac{ay}{\sqrt{x^2 + y^2}} + b \end{aligned}$$

which models the flight of a bird heading toward the origin at constant speed a , that is moved off course by a steady wind of velocity b . Determine the conditions on a and b to ensure that the solution starting at $(p, 0)$ reaches the origin. Hint: change to polar coordinates and study the phase portrait of the differential equation on the cylinder. [Chicone, *Ordinary Differential Equations with Applications*, Springer 1999, p. 86.]

H4. Proposals for term projects are due Oct. 22. Meet with me briefly to discuss your project. Bring a one paragraph proposal of the topic you will write about for my approval.

In your term paper, you will discuss some current or historical theory or application of ordinary differential equations that is not covered in the course, but it should be at the level of the course. It should be a five page mathematical paper written at a level appropriate for Math 5410 students. It should focus on one system of equations. If it is a theoretical or historical paper, it should contain some theorems with proofs that describe some phenomenon. If it is an applied paper, it should develop the equations from first principles, analyze them using tools from the course and draw conclusions about the application from the analysis. The paper should be written in proper English style, following AMS, APA,

MLA, or other recognized guidelines. Please get in touch with me if you'd like advice on your topic.

Your proposal should be a one paragraph description of what you will be writing about. In addition to a description of the differential equation and what you want to say about it, please include a reference to the topic from a book or scholarly article and a reference from the internet. Please include the URL of any website.

I will approve any reasonable proposal. The main reason I have objected is to proposals is that they propose to do more than is possible in five pages, which is a very short paper.

Please hand in problems I1–I4 on Friday, Oct. 30.

I1. Exercises from the text by Hirsch, Smale & Devaney:

184[3, 6, 8]

I2. Prove the Generalized Gronwall Inequality: Suppose $a(t)$, $b(t)$ and $u(t)$ are continuous functions defined for $0 \leq t < \infty$ and that $b(t) \geq 0$ for all $t \geq 0$. Suppose that

$$u(t) \leq a(t) + \int_0^t b(s) u(s) ds, \quad \text{for all } t \geq 0.$$

Show that

$$u(t) \leq a(t) + \int_0^t a(s) b(s) \exp\left(\int_s^t b(\tau) d\tau\right) ds, \quad \text{for all } t \geq 0.$$

I3. Find a sharp estimate for the difference in values and derivatives at T of the solutions for the two initial value problems, where u_0, u_1, ϵ are constants.

$$\begin{aligned} \ddot{x} + x &= 0, & \ddot{y} + (1 + \epsilon \sin(3t))y &= 0, \\ x(0) &= u_0, & y(0) &= u_0, \\ \dot{x}(0) &= u_1; & \dot{y}(0) &= u_1. \end{aligned}$$

I4. Linearized Stability of Fixed Points.

The SIR model of epidemics of Brauer and Castillo-Chávez relates three populations, $S(t)$ the susceptible population, $I(t)$ the infected population and $R(t)$ the recovered population. The other variables are positive constants. Assume that births in the susceptible group occur at a constant rate μK . Assume that there is a death rate of $-\mu$ for each population. Assume also that there is an infection rate of people in the susceptible population who become infected which is proportional to the contacts between the two groups βSI . There is a recovery of γI from the infected group into the recovered group. Finally, the disease is fatal to some in the infected group, which results in the removal rate $-\alpha I$ from the infected population. The resulting system of ODE's is

$$\begin{aligned} \dot{S} &= \mu K - \beta SI - \mu S \\ \dot{I} &= \beta SI - \gamma I - \mu I - \alpha I \\ \dot{R} &= \gamma I - \mu R \end{aligned}$$

1. Note that the first two equations decouple and can be treated as a 2×2 system. Then the third equation can be solved knowing $I(t)$. Let $\delta = \alpha + \gamma + \mu$. For the 2×2 system, find the nullclines and the fixed points.

2. Check the stability of the nonnegative fixed points. Show that for $\beta K < \delta$ the disease dies out. Sketch the nullclines and some trajectories in the phase plane in this case.
3. Show that for $\beta K > \delta$ the epidemic reaches a steady state. Sketch the nullclines and some trajectories in the phase plane now.

[From R. C. Robinson, *An Introduction to Dynamical Systems*, Pearson 2004.]

Please hand in problems J1, J2 on Friday, Nov. 4.

J1. Exercises from the text by Hirsch, Smale & Devaney:

210[1(any two of a-e), 2, 4, 6]

J2. Determine the stability types at the origin for the following systems.

$$(a.) \quad \begin{cases} x' = -x^3 + xy^2 \\ y' = -2x^2y - y^3 \end{cases} \qquad (b.) \quad \begin{cases} x' = -x^3 + 2y^3 \\ y' = -2xy^2 \end{cases}$$

$$(c.) \quad \begin{cases} x' = x^3 - y^3 \\ y' = xy^2 + 2x^2y + y^3 \end{cases}$$

[J. Hale and H. Koçak, *Dynamics and Bifurcations*, Springer 1991, p. 285.]

Please hand in problems K1 on Monday, Nov. 11.

K1. Exercises from the text by Hirsch, Smale & Devaney:

210[7(any three), 8(any three), 11]
229[1(any two), 2, 4]

Please hand in problems L1 – L2 on Friday, Nov.18.

L1. Exercises from the text by Hirsch, Smale & Devaney:

229[7, 10, 16]
255[10]

L2. Show that the system has a nontrivial periodic orbit.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y(9 - 4x^2 - y^2) \end{aligned}$$

Please hand in problems M1 on Friday, Dec.2.

M1. Exercises from the text by Hirsch, Smale & Devaney:

301[3, 4, 5, 6, 9, 10]

Please hand in Term Project on Wednesday, Dec. 7.

- The paper should be five pages (in some reasonable font and font size) double-spaced and printed out on paper. It should be written in good technical English. It should be written for an audience of Math 5410 students.
- There must be some mathematics, and mathematical explanation, in your paper. Just how you incorporate some mathematical exposition will vary from subject to subject. Include displayed equations and diagrams if appropriate.
- You must draw on a bare minimum of three book and journal sources. You may use blogs and Wiki articles provided that you give them credit. But also track down the source cited in a Wikipedia article.
- Give credit where it is due: whenever you use another author's ideas, whether appearing in your paper as direct quotation, paraphrase, or simply influence, you must cite them (with a footnote and then include in the bibliography). Formatting these citations and bibliography entries should be unambiguous, according to your chosen style guide.
- Please attach your essay proposal from last week to your paper.

The **FINAL** for Math 5410 - 1 is Tue., Dec. 13, 8:00 - 10:00 AM.

Half of the final will concentrate on the last third of the material covered: 11.2 and 13.1 - 13.8. The other half will be comprehensive. You will be allowed to bring a cheat sheet, an 8.5" x 11" paper with notes on both sides.