Math 5410 § 1.	**First Midterm Exam**	Name:______________
Treibergs		Sept. 23, 2020

1. *For the system, Sketch the regions in the ab-plane where this system has different types of canonical forms. In the interior of each region, sketch a small phase plane indicating how the flow looks.*

$$X' = \begin{pmatrix} a & 1 \\ b & a \end{pmatrix} X.$$

The eigenvalues satisfy

$$\begin{vmatrix} a-\lambda & 1 \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b = 0$$

so $\lambda = a \pm \sqrt{b}$.

If $b < 0$ then spiral which is stable if also $a < 0$, center if also $a = 0$ and unstable if also $a > 0$.

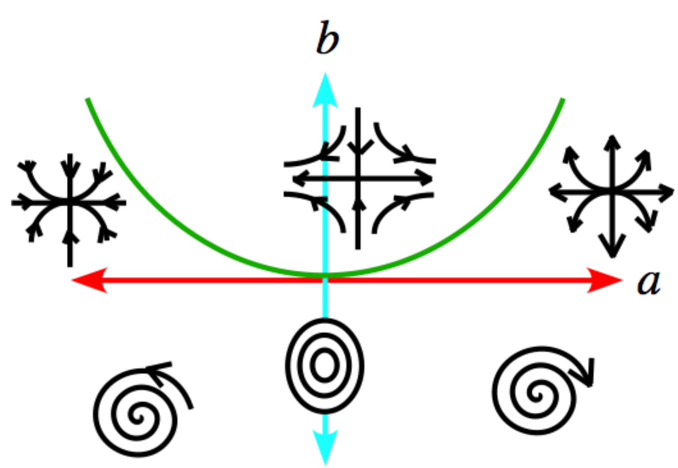
If $b = 0$ then $\lambda = a, a$ which are proper nodes, stable if also $a < 0$ and unstable if also $a > 0$.

If $0 < b < a^2$ improper node which are stable if also $a < 0$ and unstable if also $a > 0$.

If $b = a^2$ then roots are 0 and $2a$, which are "combs," incoming if $a < 0$ and outgoing if $a > 0$.

If $b > a^2$ then $\lambda_1 = a - \sqrt{b} < 0 < a + \sqrt{b} = \lambda_2$ are saddles.

The cases are summarized in the sketch.



Sketch of flows in interiors of the regions in the (a, b) plane.

2. *For the system Find the real valued general solution. Determine the canonical form for this equation. Find the matrix P so that $Y = PX$ puts (1) in canonical form. Check that your matrix works.*

$$X' = \begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix} X. \tag{1}$$

The eigenvalues satisfy

$$0 = \begin{vmatrix} 5-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) + 4 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

so $\lambda = 3, 3$. Computing eigenvectors, one sees for $\lambda = 3$,

$$0 = (A - \lambda I)v = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$A - \lambda I$ has rank two so there is no other independent eigenvector. Solving for a cyclic vector, we find

$$(A - \lambda I)w = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v$$

Thus the transformation $x = Py$ to reduce to canonical form $\dot{y} = Jy$ is

$$P = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

We check that $AP = PJ$

$$AP = \begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & \frac{5}{2} \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = PJ$$

3. *For the system find the real valued general solution. Find the flow generated by this system, $\varphi_t(c)$, which gives the point on the trajectory starting at c at the time t.*

$$X' = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} X.$$

The eigenvalues satisfy

$$0 = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 5 = (\lambda - 2)^2 + 1$$

so $\lambda = 2 \pm i$. For $\lambda = 2 + i$, computing an eigenvector

$$0 = (A - \lambda I)v = \begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix} \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

The corresponding complex solution is

$$e^{(2+i)t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{2t}(\cos t + i \sin t) \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{2t} \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + ie^{2t} \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}$$

Taking real and imaginary parts as a basis, the general solution is

$$\phi_t(c) = k_1 e^{2t} \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + k_2 e^{2t} \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}$$

where at $t = 0$ we have $\phi_0(c) = c$ so

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \phi_0(c) = \begin{pmatrix} k_1 \\ k_1 + k_2 \end{pmatrix}$$

Thus $k_1 = c_1$ and $k_2 = c_2 - c_1$. Thus the flow is $\phi_t(c) =$

$$c_1 e^{2t} \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + (c_2 - c_1)e^{2t} \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix} = e^{2t} \begin{pmatrix} \cos t - \sin t & \sin t \\ -2\sin t & \cos t + \sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

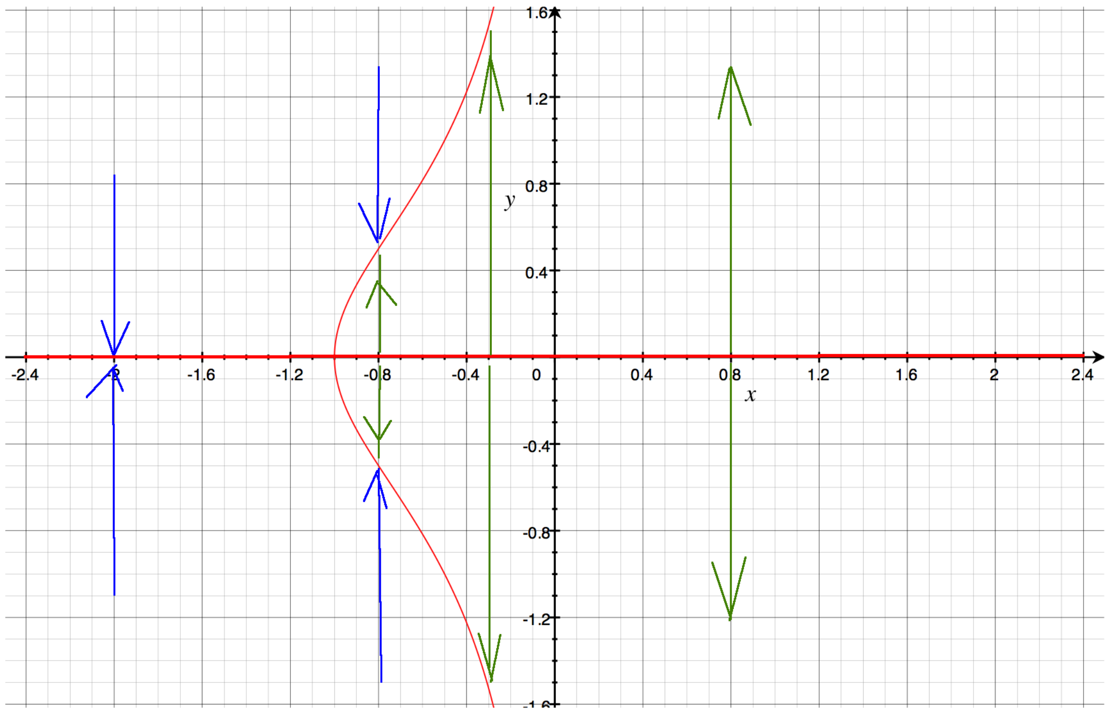
4. *Consider the family of differential equations depending on the parameter a. Find a bifurcation points. Sketch the bifurcation diagram for this family of equations. Sketch the phase lines for values of a above and below the bifurcation values. Are any of the flows in these regions topologically conjugate? Explain.*

$$x' = \frac{x}{1+x^2} + ax$$

The rest points satisfy

$$0 = x\left(\frac{1}{1+x^2} + a\right)$$

so they are the curves $x = 0$ and $a = -\frac{1}{1+x^2}$ in the (a, x) plane. There are two bifurcation points, at $a = -1$ and $a = 0$. For $a < -1$ 0 is the only rest point, and it is stable. There is a pitchfork bifurcation at $(a, x) = (-1, 0)$. As a increases, the single rest point splits into three, two stable ones at $x = \pm\sqrt{a^{-1} - 1}$ and an unstable one at $x = 0$. Then another bifurcation occurs at $a = 0$ where the stable rest points disappear, leaving only the unstable one at $x = 0$. The phase lines have been skecthe vertically for several typical a's. None of the flows in each of the three regions $a < -1$, $-1 < a < 0$ and $0 < a$ are not topologically conjugate. There is only one stable rest point in the first, there are three rest points in the second, and there is only on unstable rest point in the third.



Bifurcation curves.

5. *Find the flows ϕ_t^X and ϕ_t^Y. Find an explicit topological congugacy between the flows.*

$$X' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} X, \qquad Y' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} Y.$$

The eigenvalues for X satisfy

$$0 = \begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = -\lambda(3-\lambda) + 2 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

so $\lambda = 1, 2$. For $\lambda_i = 1, 2$, computing an eigenvector

$$0 = (A - \lambda_1 I)v = \begin{pmatrix} -2 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \qquad 0 = (A - \lambda_2 I)v = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix};$$

Hence $X = PY$ converts the first equation to $\dot{Y} = JY$ where

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad P^{-1}AP = J = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Thus Y satisfies the second given equation. The general solution is

$$X(t) = k_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = x(0) = \begin{pmatrix} k_1 + k_2 \\ k_1 + 2k_2 \end{pmatrix}$$

so $k_1 = 2c_1 - c_2$ and $k_2 = c_2 - c_1$. Thus, the flow is

$$\phi_t^X(c) = (2c_1 - c_2)e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c_2 - c_1)e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2e^t - e^{2t} & e^{2t} - e^t \\ 2e^t - 2e^{2t} & 2e^{2t} - e^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Since the matrix is diagonal, the second equation decouples and the general solution is

$$\phi_t^Y(c) = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The linear map that gave us the reduction to normal form also gives the topological conjugacy. Thus

$$Y = h(X) = P^{-1}X = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} X.$$

The problem did not ask us to check if this indeed gives the conjugacy. But in the matrix formulation where maps are given by matrix multiplication, this is easy to check.

$$\phi_t^Y \circ h = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2e^t & -e^t \\ -e^{2t} & e^{2t} \end{pmatrix}$$

$$h \circ \phi_t^X = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2e^t - e^{2t} & e^{2t} - e^t \\ 2e^t - 2e^{2t} & 2e^{2t} - e^t \end{pmatrix} = \begin{pmatrix} 2e^t & -e^t \\ -e^{2t} & e^{2t} \end{pmatrix}$$

Since they are the same, the map h gives a topological conjugacy.