Math 5410 § 1. **Half of Final Exam** Name: Practice Problems

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**Half of the final will be over material since the last midterm exam, such as the practice problems given here. The other half will be comprehensive.**

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1. *Consider this enhanced predator prey system for the populations of two species $x, y \ge 0$. Find the nullclines and use this information to sketch the phase space. Indicate the equilibrium points and the direction of flow. Determine the stability types of the rest points and find the stable and unstable spaces of the linearized flows at these points. Sketch enough trajectories to describe the global flow.*

$$\begin{aligned} x' &= x(-1+y) \\ y' &= y(3-x-y) \end{aligned}$$

$x' = 0$ if $x = 0$ or $y = 1$. $y' = 0$ if $y = 0$ or $x + y = 3$. Thus both equations are satisfied at the equilibrium points $P_1 = (0,0)$, $P_2 = (0,3)$ and $P_3 = (2,1)$. We note that $x' > 0$ if $y < 1$ and $x' < 0$ if $y > 1$. Similarly $y' > 0$ if $x + y < 3$ and $y' < 0$ if $x + y > 3$. Thus in the four regions $NE$, $NW$, $SW$ and $SE$ cut by $y = 1$ and $x + y = 3$ going clockwise around $P_3$, the direction of flow is $SE$, $NE$, $NW$ ande $SW$, respectively.

The Jacobean is

$$DF(x,y) = \begin{pmatrix} y-1 & x \\ -y & 3-2y-x \end{pmatrix}.$$

At $P_1$ we have

$$DF(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

so that the eigenvalues and eigenvectors are $\lambda_1 = -1$, $V_1 = (1,0)$; $\lambda_2 = 3$, $V_2 = (0,1)$. This is a saddle with $E^s$ and $E^u$ the horizontal and vertical axes, resp.

At $P_2$ we have

$$DF(0,3) = \begin{pmatrix} 2 & 0 \\ -3 & -3 \end{pmatrix}$$

so that the eigenvalues and eigenvectors are $\lambda_1 = 2$, $V_1 = (5,-3)$; $\lambda_2 = -3$, $V_2 = (0,1)$. This is a saddle with $E^s$ the vertical axis and $E^u = \operatorname{span}\{V_1\}$.

At $P_3$ we have

$$DF(2,1) = \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix}.$$

The trace is negative and the determinant is positive, so that the eigenvalues are both negative or have negative real parts. Indeed the eigenvalues are $\lambda_1 = \frac{1}{2}(-1 + \sqrt{7}i)$. This is a stable spiral.

All of this information is given in Fig. 1. The computed phase plane generated by 3D-XplorMath is given in Fig. 2.

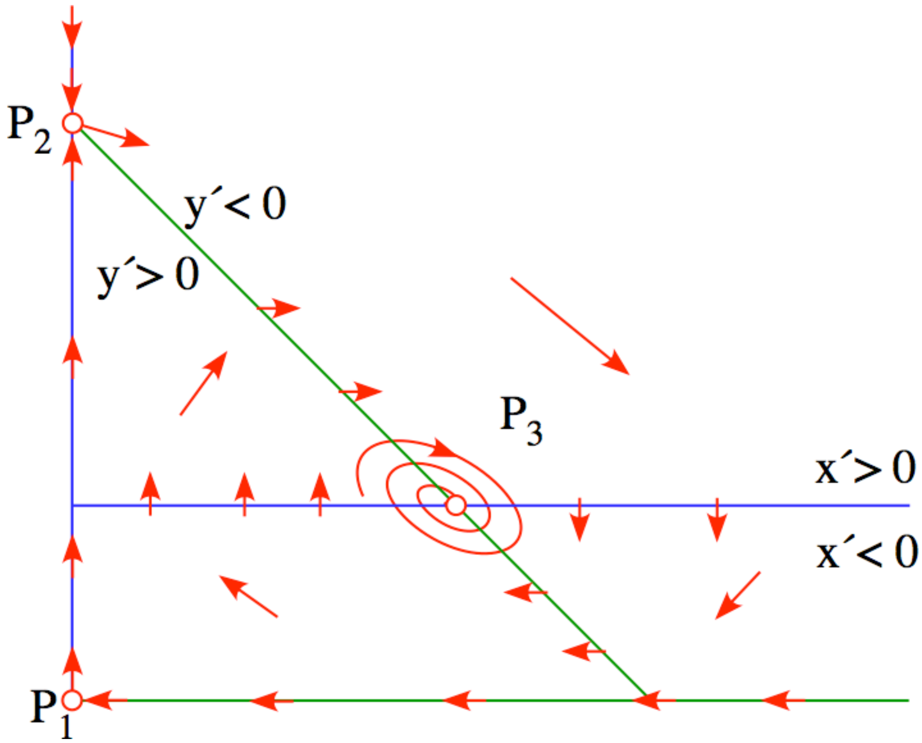


Figure 1: Nullclines, Equilibrium Points, Directions and Stable/Unstable Spaces.

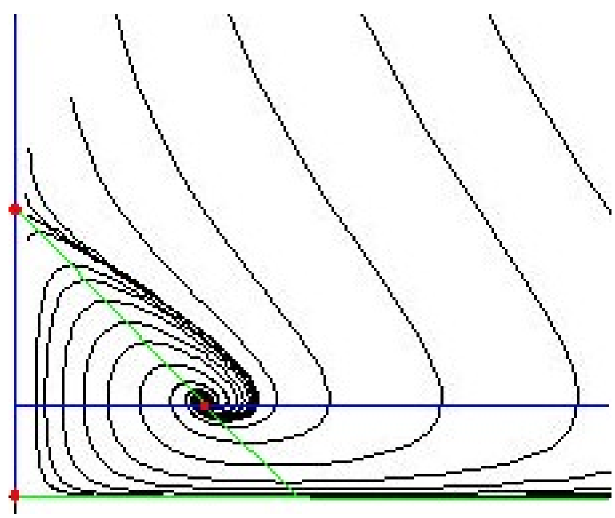

Figure 2: Trajectories of the predator prey system of Problem 1..

2. *Consider the folowing model of reaction diffusion equation system. Determine if it is a gradient or a Hamiltonian system. Use this fact to sketch the global flow. Identify the stability type of the equilibrium points.*

$$\begin{aligned} x' &= 2(y-x) + x(1-x^2) \\ y' &= -2(y-x) + y(1-y^2) \end{aligned}$$

We see that $f_y(x,y) = 2$ and $g_x(x,y) = 2$ are equal so there is a potential function

$$F(x,y) = \frac{1}{2}x^2 + \frac{1}{4}x^4 - 2xy + \frac{1}{2}y^2 + \frac{1}{4}x^4$$

and the system is a gradient system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = -\nabla F(x,y).$$

The critical points satisfy

$$\begin{aligned} 0 = f(x,y) &= -F_x = 2y - x - x^3 \\ 0 = g(x,y) &= -Fy = 2x - y - y^3 \end{aligned}$$

Thus the equilibrium points are the intersections of the curves $y = \frac{1}{2}(x + x^3)$ and $y = \frac{1}{2}(y + y^3)$, which are the points $P_1 = (0,0)$, $P_2 = (-1,-1)$ and $P_3 = (1,1)$. Computing the Hessian,

$$H(x,y) = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix} = \begin{pmatrix} -1 - 3x^3 & 2 \\ 2 & -1 - 3y^2 \end{pmatrix}$$

At $P_1$,

$$H(0,0) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -6$, hence $P_2$ and $P_3$ are relative minima.

At $P_2$ and $P_3$,

$$H(-1,-1) = H(1,1) = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -3$, hence is a saddle.

It follows that $P_1$ is a saddle for the flow and both $P_2$ and $P_3$ are stable nodes. The flow is perpendicular to the level surfaces. The 3D-XplorMath plot of level sets and trajectories is given in Fig. 4.

3. *Suppose that $f(x,t) \in \mathcal{C}^(\mathbf{R}^2, \mathbf{R}^1)$ and that $(t_0, x_0) \in \mathbf{R}^2$. Assume that $f$ satisfies a local Lipschitz Condition: for every compact set $K \subset \mathbf{R}^2$ there is $L \in \mathbf{R}$ so that*

$$|f(t,x) - f(t,y)| \le L|x-y| \qquad \textit{whenever } (t,x),(t,y) \in \mathbf{K}.$$

*Then there is an $\epsilon > 0$ and a function $x(t) \in \mathcal{C}^1([t_0, t_0+\epsilon), \mathbf{R}^2)$ which solves the initial value problem*

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0. \end{aligned} \tag{1}$$

This is the short time existence theorem for non-autonomous equations. Note that we only assume continuity with respect to $t$ so that we cannot consider time as an independent variable in a system of one higher dimension and quote the result we proved in class for autonomous equations. However, we may follow the proof we gave verbatim.

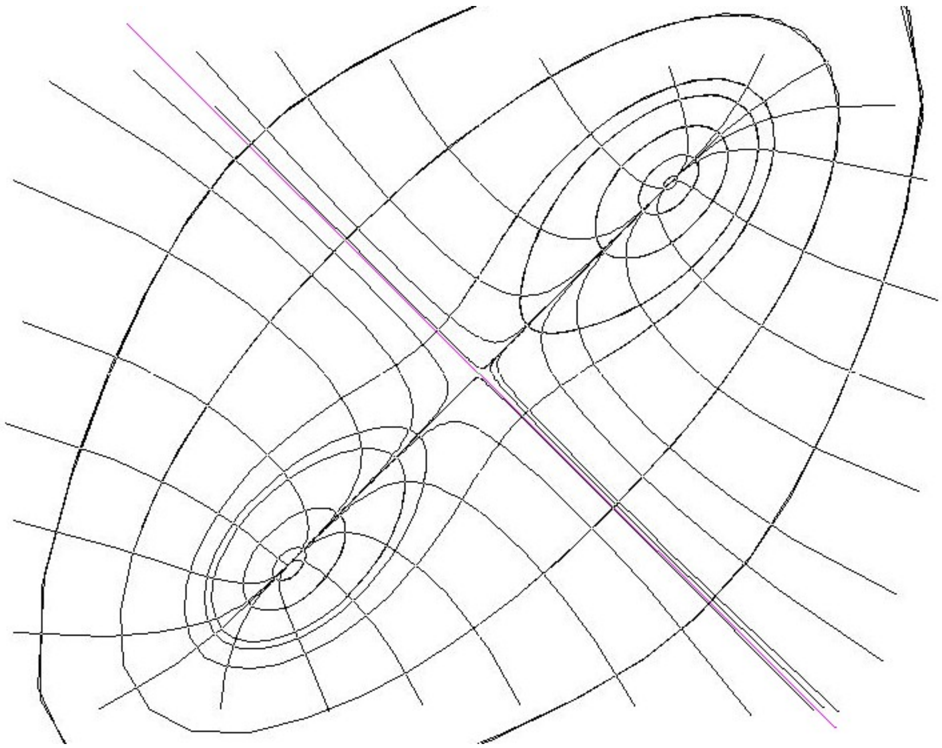

Figure 3: Flow of gradient system and level curves of the potential.

For any $r > 0$ let $R = [t_0, t_0 + r] \times [x_0 - r, x_0 + r]$ be a compact compact box and $L$ its corresponding Lipschitz constant. Let

$$M = \sup_{(t,x) \in R} |f(t, x)|.$$

Let $\epsilon = \min\left\{r, \frac{M}{r}, \frac{1}{L+1}\right\}$. Let

$$\mathcal{X} = \{z(t) \in \mathcal{C}([t_0, t_0 + \epsilon]) : |z(t) - x_0| \leq r \text{ whenever } t \in [t_0, t_0 + \epsilon]\}$$

The space $\mathcal{X}$ is a closed convex subset of the complete metric space $C([t_0, t_0 + \epsilon])$ under the sup norm. The sup norm for $w \in \mathcal{C}([t_0, t_0 + \epsilon])$ is given by

$$\|w\| = \sup_{s \in [t_0, t_0 + \epsilon]} |\, w(s)\,|.$$

Thus the subspace $(\mathcal{X}, \| \bullet \|)$ is also a complete metric space.

We will show that there is a unique function $z \in \mathcal{X}$ that solves the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \qquad \text{for all } t \in [t_0, t_0 + \epsilon] \tag{2}$$

Define the nonlinear operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ for $u \in \mathcal{X}$ and $t \in [t_0, t_0 + \epsilon]$ by the formula

$$(\mathcal{T}u)(t) = x_0 + \int_{t_0}^{t} f(s, u(s))\, ds.$$

Note that $\mathcal{T}u$ is continuous since it is the indefinite integral of a continuous function $s \mapsto f(s, u(s))$. To see that is satisfies the bound, we have for any for $u \in \mathcal{X}$ and $t \in [t_0, t_0 + \epsilon]$

$$\begin{aligned} |(\mathcal{T}u)(t) - x_0| &= \left| \int_{t_0}^{t} f(s, u(s))\, ds \right| \le \int_{t_0}^{t} |f(s, u(s))|\, ds \\ &\le \int_{t_0}^{t} M\, ds = M(t - t_0) \le M\epsilon \le M \cdot \frac{r}{M} \le r. \end{aligned}$$

Thus we have shown that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$.

Now we shall show that the operator is a contraction, namely, for some $\theta \in (0, 1)$ we have for all $u, v \in \mathcal{X}$

$$\|\mathcal{T}u - \mathcal{T}v\| \le \theta \|u - v\|. \tag{3}$$

In this situation, we define the Picard Iterates recursively: $x_0(t) = x_0$ and $x_{n+1}(t) = (\mathcal{T}x_n)(t)$ for $n \in \mathbf{N}$. If the mapping $\mathcal{T}$ is a contraction, then the Picard Iterates $\{x_n(t)\}$ are dominated by a geometric sequence, thus form a Uniformly Cauchy sequence on $[t_0, t_0 + \epsilon]$, thus converges uniformly to a unique continuous function in the complete metric space $(\mathcal{X}, \| \bullet \|)$ satisfying $x = \mathcal{T}x$, which is the integral equation (2). This fact from analysis is called the *Contraction Mapping Theorem* or the *Banach Fixed Point Theorem.* For $u, v \in \mathcal{X}$ and $t \in [t_0, t_0 + \epsilon]$ we have by the Lipschitz bound,

$$\begin{aligned} |(\mathcal{T}u)(t) - (\mathcal{T}v)(t)| &= \left| \int_{t_0}^{t} f(s, u(s)) - f(s, v(s))\, ds \right| \\ &\le \int_{t_0}^{t} |f(s, u(s)) - f(s, v(s))|\, ds \\ &\le \int_{t_0}^{t} L|u(s) - v(s)|\, ds \\ &\le \int_{t_0}^{t} L\|u - v\|\, ds = (t - t_0)\|u - v\| \le \frac{L}{L+1}\|u - v\|. \end{aligned}$$

Taking supremum over $t \in [t_0, t_0 + \epsilon]$ gives (3) with $\theta = \dfrac{L}{L+1}$.

It remains to argue that the solution of the integral equation (2) also solves the initial value problem (1). If $x(t)$ solves (1), then by the Fundamental Theorem of Calculus, for $t \in [t_0, t_0 + \epsilon]$,

$$x(t) - x_0 = \int_{t_0}^{t} \frac{dx}{dt}(s)\, ds = \int_{t_0}^{t} f(s, x(s))\, ds$$

which is the integral equation (2). On the other hand, if $x(t) \in \mathcal{X}$ solves the integral equation (2), then it is differentiable since by the Fundamental Theorem of Calculus, $x(x)$ is the integral of a continuous function. Moreover, its derivative is given by the integrand

$$\frac{dx}{dt}(t) = f(t, x(t)).$$

Also, at the initial point

$$x(t_0) = x_0 + \int_{t_0}^{t_0} f(s, x(s))\, ds = x_0 + 0,$$

so the initial condition holds as well. □

4. *Suppose that* $f(x,t) \in \mathcal{C}(\mathbf{R}^2, \mathbf{R}^1)$*,* $(t_0, x_0) \in \mathbf{R}^2$ *and* $a > 0$*. Assume that* $f$ *satisfies a Lipschitz Condition: there is* $L \in \mathbf{R}$ *so that*

$$|f(t,x) - f(t,y)| \le L|x-y| \qquad \textit{whenever } (t,x),(t,y) \in [t_0, t_0+a] \times \mathbf{R}.$$

*Show that the soliution of the initial value problem (1) is unique.*

Suppose that both $u(t)$ and $v(t)$ satisfy the initial value problem (1). Using ther integral equation, we find

$$\begin{aligned} |u(t) - v(t)| &= \left| \int_{t_0}^{t} f(s,u(s)) - f(s,v(s))\, ds \right| \\ &\le \int_{t_0}^{t} |f(s,u(s)) - f(s,v(s))|\, ds \\ &\le \int_{t_0}^{t} L|u(s) - v(s))|\, ds \end{aligned}$$

Thus if we define the function

$$F(t) = \int_{t_0}^{t} L|u(s) - v(s))|\, ds$$

we find fromt the Fundamental Theorem of Calculus that

$$F'(t) = |u(t) - v(t))| \le LF(t).$$

It follows that

$$\left(e^{-L(t-t_0)} F\right)' = e^{-L(t-t_0)}(F' - LF) \le 0.$$

Thus the positive function satisfies

$$0 \le F(t) \le F(t_0) = 0$$

for all $t$. Thus $F(t) = 0$ or $u(t) = v(t)$ for all $t$. □

5. *Generalize the method of Picard Approximation to accommodate non-autonomous equtions. Find the first few iterates of the initial value problem and compare to the Maclaurin series for the solution.*

$$\begin{aligned} x' &= -2tx \\ x(0) &= 1. \end{aligned}$$

The generalization is to use the non-autonomous integral equation. Put $x_0(t) = x_0$ or anything in $\mathcal{X}$. Then recursively devine for $n \in \mathbf{N}$,

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

In this example, let $x_0(t) = 1$. Then

$$\begin{aligned} x_1(t) &= x_0 - \int_0^t 2s x_0(s)\, ds = 1 - 2\int_0^t s\, ds = 1 - t^2; \\ x_2(t) &= x_0 - \int_0^t 2s x_1(s)\, ds = 1 - 2\int_0^t s\left(1 - s^2\right)\, ds = 1 - t^2 + \frac{1}{2}t^4; \\ x_3(t) &= x_0 - \int_0^t 2s x_2(s)\, ds = 1 - 2\int_0^t s\left(1 - s^2 + \frac{1}{2}s^4\right)\, ds = 1 - t^2 + \frac{1}{2}t^4 - \frac{1}{6}t^6; \\ x_4(t) &= x_0 - \int_0^t 2s x_3(s)\, ds = 1 - 2\int_0^t s\left(1 - s^2 + \frac{1}{2}s^4 - \frac{1}{6}s^6\right)\, ds = 1 - t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 - \frac{1}{24}t^8; \end{aligned}$$

The Maclaurin series solution is found by assuming the solutin is a convergent power series and plugging into the ODE to find equations for the coefficients. Assuming

$$x(t) = 1 + \sum_{i=1}^{\infty} a_n t^n$$

we get

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^4 + \cdots$$

and

$$-2tx = -2t - \sum_{n=1}^{\infty} 2a_n t^{n+1} = -2t - 2a_1 t^2 - 2a_2 t^3 - 2a_3 t^4 - \cdots$$

which results in the equations

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 0, \\ a_2 &= -1, \\ n a_n &= -2a_{n-2}, \qquad \text{for } n \geq 3. \end{aligned}$$

This results in $a_{2k-1} = 0$ for all odd terms and

$$a_{2k} = -\frac{2}{2k} a_{2(k-1)}$$

whose solution is $a_4 = \frac{1}{2}$, $a_6 = -\frac{1}{6}$, $a_8 = \frac{1}{24}$ and so on. In general, for $k \in \mathbf{N}$ we have

$$a_{2k} = \frac{(-1)^k}{k!}.$$

In other words $x(t) = e^{-t^2}$. □

6. *Suppose that both $f(t,x), g(x,t) \in \mathcal{C}(\mathbf{R}^2, \mathbf{R}^1)$, $(t_0, x_0) \in \mathbf{R}^2$ and $a > 0$ satisfies the Lipschitz Condition: there is $L \in \mathbf{R}$ so that*

$$|f(t,x) - f(t,y)| \leq L|x-y| \qquad \text{whenever } (t,x),(t,y) \in [t_0, t_0 + a] \times \mathbf{R}.$$

*Show that the soliution of the initial value problem (1) depends continuously on the equation. That is, assume that for some $\eta > 0$ the two functions satisfy*

$$|f(t,x) - g(t,x)| \leq \eta \qquad \text{for all } (t,x) \in [t_0, t_0 + a] \times \mathbf{R}.$$

*Suppose we look at two solutions of the different equations*

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0. \end{cases}, \qquad \begin{cases} y' = g(t,y) \\ y(t_0) = x_0. \end{cases}$$

*Then*

$$|x(t) - y(t)| \leq \frac{\eta}{L}\left(e^{L(t-t_0)} - 1\right) \qquad \text{for all } t \in [t_0, t_0 + a].$$

As usual, we will try to apply Gronwall's Inequality to the difference of solutions. For $t \in [t_0, t_0 + a]$,

$$
\begin{aligned}
|x(t) - y(t)| &= \left| x_0 + \int_{t_0}^{t} f(s, x(s))\, ds - x_0 - \int_{t_0}^{t} g(s, y(s))\, ds \right| \\
&= \left| \int_{t_0}^{t} f(s, x(s)) - g(s, y(s))\, ds \right| \\
&\leq \int_{t_0}^{t} |f(s, x(s)) - g(s, y(s))|\, ds \\
&\leq \int_{t_0}^{t} |f(s, x(s)) - g(s, x(s)) + g(s, x(s)) - g(s, y(s))|\, ds \\
&\leq \int_{t_0}^{t} |f(s, x(s)) - g(s, x(s))| + |g(s, x(s)) - g(s, y(s))|\, ds \\
&\leq \int_{t_0}^{t} \eta + L|x(s) - y(s)|\, ds \\
&= \eta(t - t_0) + \int_{t_0}^{t} L|x(s) - y(s)|\, ds
\end{aligned}
$$

We apply Gronwall's Inequality (Problem 5) with $f(t) = \eta(t - t_0)$, $u = |x(t) - y(t)|$ and $v = L$. For $t \in [t_0, t_0 + a]$,

$$
\begin{aligned}
|x(t) - y(t)| &\leq \eta(t - t_0) + \eta L \int_{t_0}^{t} e^{L(t-s)}(s - t_0)\, ds \\
&= \frac{\eta}{L}\left(e^{L(t-t_0)} - 1\right).
\end{aligned}
$$

7. *Prove Gronwall's Inequality. If $f, u, v : [t_0, t_0 + a] \to \mathbf{R}$ are continuous, $u \geq 0$, $v \geq 0$ and satisfy*

$$u(t) \leq f(t) + \int_{t_0}^{t} u(s)\, v(s)\, ds \qquad \text{for all } t$$

then

$$u(t) \leq f(t) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} v(u)\, du\right) f(s) v(s)\, ds.$$

Let

$$F(t) = \int_{t_0}^{t} u(s)\, v(s)\, ds.$$

Then

$$F' = uv \leq v(f + F)$$

so

$$\left(e^{-\int_{t_0}^{t} v(s)\, ds} F\right)' = e^{-\int_{t_0}^{t} v(s)\, ds}(F' - vF) \leq e^{-\int_{t_0}^{t} v(s)\, ds} v f$$

which implies

$$e^{-\int_{t_0}^{t} v(r)\, dr} F(t) \leq \int_{t_0}^{t} e^{-\int_{t_0}^{s} v(r)\, dr} v(s)\, f(s)\, ds.$$

Multiplying by the integrating factor yields

$$u(t) \leq f(t) + \int_{t_0}^{t} e^{\int_{s}^{t} v(r)\, dr} v(s)\, f(s)\, ds. \quad \square$$

8. *Analyze the stability properties of each rest point. At each rest point, determine the dimensions of the stable and unstable manifolds. Compute a basis for the stable and unstable manifolds for the associated linearized system.*

$$3x''' - 7x'' + 3x' + e^x - 1 = 0$$

We recast the equation as a $3 \times 3$ system.

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= \frac{1}{3}(1 - e^x) - y + \frac{7}{3}z \end{aligned}$$

The equilibrium point satisfies $y = 0$, $z = 0$ so $0 = 1 - e^x$ so $x = 0$. The Jacobean at $(0, 0, 0)$ is

$$A = DF(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & -1 & \frac{7}{3} \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = (-\lambda)(-\lambda)\left(\frac{7}{3} - \lambda\right) - \frac{1}{3} + (-\lambda) = -\lambda^3 + \frac{7}{3}\lambda^2 - \lambda - \frac{1}{3} = -(\lambda - 1)\left(\lambda^2 - \frac{4}{3}\lambda - \frac{1}{3}\right)$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \frac{2}{3} + \frac{\sqrt{7}}{3}$ and $\lambda_3 = \frac{2}{3} - \frac{\sqrt{7}}{3}$. Thus the matrix is hyperbolic with two positive and one negative eigenvalue. The origin is of saddle type, thus unstable. It has a local one dimensional stable manifold and a two dimensional local unstable manifold. We solve for eigenvectors.

$$0 = (A - \lambda_1 I)V_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -\frac{1}{3} & -1 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$0 = (A - \lambda_2 I)V_2 = \begin{pmatrix} -\frac{2}{3} - \frac{\sqrt{7}}{3} & 1 & 0 \\ 0 & -\frac{2}{3} - \frac{\sqrt{7}}{3} & 1 \\ -\frac{1}{3} & -1 & \frac{5}{3} - \frac{\sqrt{7}}{3} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{2}{3} + \frac{\sqrt{7}}{3} \\ \frac{11}{9} + \frac{4\sqrt{7}}{9} \end{pmatrix}$$

$$0 = (A - \lambda_3 I)V_3 = \begin{pmatrix} -\frac{2}{3} + \frac{\sqrt{7}}{3} & 1 & 0 \\ 0 & -\frac{2}{3} + \frac{\sqrt{7}}{3} & 1 \\ -\frac{1}{3} & -1 & \frac{5}{3} + \frac{\sqrt{7}}{3} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{2}{3} - \frac{\sqrt{7}}{3} \\ \frac{11}{9} - \frac{4\sqrt{7}}{9} \end{pmatrix}$$

It follows that $V_1$ and $V_2$ span the unstable space $E^u$ of the linearied equation $y' = Ay$ and $V_3$ spans the stable space $E^s$.

9. *Find the stable and unstable manifolds.*

$$
\begin{aligned}
x' &= -x \\
y' &= -y + x^2 \\
z' &= z + y^2
\end{aligned}
$$

The only equilibrium point is the origin. There the linearized system is $A = DF(0) = \operatorname{diag}(-1, -1, 1)$ so that the stable and unstable spaces are $E^s = \{(x, y, z) : z = 0\}$ and $E^u = \{(x, y, z) : x = y = 0\}$. To find the stable manifold, we seek solutions that are tangent to $E^s$ at the origin. The system may be solved by back substitution. Indeed, $x' = -x$ means

$$x(t) = x_0 e^{-t}.$$

Hence $y' = -y + x^2 = -y + x_0^2 e^{-2t}$ so

$$y(t) = \left(y_0 + x_0^2\right) e^{-t} - x_0^2 e^{-2t}$$

Also $z' = z + y^2 = z + \left(y_0 + x_0^2\right)^2 e^{-2t} - 2x_0^2 \left(y_0 + x_0^2\right) e^{-3t} + x_0^4 e^{-4t}$ so

$$
\begin{aligned}
z(t) = &\left( z_0 + \frac{1}{3} \left(y_0 + x_0^2\right)^2 - \frac{1}{2} x_0^2 \left(y_0 + x_0^2\right) + \frac{1}{5} x_0^4 \right) e^t \\
&- \frac{1}{3} \left(y_0 + x_0^2\right)^2 e^{-2t} + \frac{1}{2} x_0^2 \left(y_0 + x_0^2\right) e^{-3t} - \frac{1}{5} x_0^4 e^{-4t}
\end{aligned}
$$

The stable manifold tends to zero as $t \to \infty$ so the $e^t$ coefficient must vanish, namely, points of $W^s$ satisfy

$$
\begin{aligned}
z_0 &= -\frac{1}{3} \left(y_0 + x_0^2\right)^2 + \frac{1}{2} x_0^2 \left(y_0 + x_0^2\right) - \frac{1}{5} x_0^4 \\
&= -\frac{1}{3} y_0^2 - \frac{1}{6} x_0^2 y_0 - \frac{1}{30} x_0^4.
\end{aligned}
$$

The points in the unstable manifold $W^u$ tend to zero as $t \to -\infty$, which means that the coefficients of the $e^{-t}$ and $e^{-2t}$, $e^{-3t}$ and $e^{-4t}$ must vanish, namely

$$x_0 = y_0 = 0.$$

10. *Classify the stability types of all the equilibrium points of the system.*

$$
\begin{aligned}
x' &= -4y + 2xy - 8 \\
y' &= 4y^2 - x^2
\end{aligned}
$$

At the equilibrium, the second equation $0 = y'$ says $x = \pm 2y$. Using $x = 2y$ in $0 = x'$ says $0 = -2y + 2y^2 - 4$ or $y = -1, 2$, giving two rest points $P_1 = (-2, -1)$ and $P_2 = (4, 2)$. Using $x = -2y$ says $y^2 + y + 2 = 0$ which has no real roots.

The Jacobean is

$$A = DF = \begin{pmatrix} 2y & -4 + 2x \\ -2x & 8y \end{pmatrix}$$

At $P_1$,

$$A = \begin{pmatrix} -2 & -8 \\ 4 & -8 \end{pmatrix}$$

whose eigenvalues are $\lambda = -5 \pm i\sqrt{23}i$, thus by the linearized stability, $P_1$ is a stable node node.

At $P_2$,

$$A = \begin{pmatrix} 4 & 4 \\ -8 & 16 \end{pmatrix}$$

whose eigenvalues are $\lambda = 2, 3$, thus by the linearized stability, $P_2$ is an unstable node.

11. *Show that there is a nontrivial periodic solution.*

$$\begin{aligned} x' &= y \\ y' &= -x + (1 - x^2 - 2y^2)y \end{aligned}$$

Observe that the origin is the only rest point. Consider the function

$$V(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

Then along the flow, in polar coordinates,

$$\dot{V}(x, y) = (1 - x^2 - 2y^2)y^2$$

Thus inside the circle $x^2+y^2 \le \frac{1}{2}$ we have $1-x^2-2y^2 \ge 1-2(x^2+y^2) \ge 0$ so that $\dot{V} \ge 0$ and the flow is not inward. Outside the circle $x^2+y^2 \ge 1$ we have $1-x^2-2y^2 \le 1-x^2-y^2 \le 0$ so that $\dot{V} \le 0$ and the flow is not outward.

It follows that the annulus $\frac{1}{2} \le r^2 \le 1$ is a closed, forward invariant set without rest points. Hence it must contain its limit cycles. Indeed, for any point in the annulus, its omega limit set $\omega(x)$ must be a nonempty, closed limit set in the annulus which contains no equilibrium point since the origin is not in the annulus. By the Poincaré Bendixson Theorem, $\omega(x)$ is a closed orbit.

12. *Consider the prameterized system. Find the bifurcation values $\mu = \mu_0$ and describe the nature of the bifurcation at the $\mu_0$.*

$$\begin{aligned} x' &= \mu x - y + x\sqrt{x^2 + y^2} \\ y' &= x + \mu y + y\sqrt{x^2 + y^2} \end{aligned}$$

Write the system in polar coordinates. With $x = r\cos\theta$ and $y = r\sin\theta$ we get

$$\begin{aligned} r'\cos\theta - r\sin(\theta)\theta' &= \mu r\cos\theta - r\sin\theta + r^2\cos\theta \\ r'\sin\theta + r\cos(\theta)\theta' &= r\cos\theta + \mu r\sin\theta + r^2\sin\theta \end{aligned}$$

so

$$\begin{aligned} \theta' &= 1 \\ r' &= \mu r + r^2 \end{aligned}$$

If $\mu \ge 0$ then $r' > 0$ all $r > 0$ so that the origin is an unstable node. If $\mu < 0$ then at the radius $r = -\mu$, there is a limit cycle $C$. In this case if $0 < r < -\mu$ we have $r' < 0$ and for $r > -\mu$, $r' > 0$. In other words, the cycle $C$ is unstable: nearby trajectories depart from $C$ as $t \to \infty$. At the same time, the origin is a stable rest point.

Thus this system undergoes a Hopf bifurcation. As $\mu$ passes through zero from negative to positive, an unstable limit cycle $C$ and a stable rest point zero collapse to form an unstable rest point zero.

13. *Determine the stability type of the zero solution.*

$$x'' + (x')^3 + x = 0$$

The corresponding system is

$$\begin{aligned} x' &= y \\ y' &= -x - y^3 \end{aligned}$$

Consider the Lyapunov function

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

We have

$$\dot{V} = xx' + yy' = -y^4.$$

Since $\dot{v} \leq 0$, the origin is stable. In fact, in any closed ball $P = \overline{B(0,\delta)}$, the zero set is $Z = \{(x,y) : \dot{V}(x,y) = 0\} = \{(x,y) : y = 0\}$. In this set there are no invariant subsets other than $\{(0,0)\}$ because, if $y = 0$ and $x \neq 0$ then $y' \neq 0$ so the trajectory leaves $Z$. By LaSalle's Principle, $(0,0)$ is asymptotically stable and $P$ is in the basin of its attraction. But as $\delta > 0$ is arbitrary, every point $(x_0, y_0) \in \mathbf{R}^2$ is attracted to the origin.