Homework for Math 5210 - 002, Spring 2025

A. Treibergs, Instructor

April 10, 2025

Our text is by N. L. Carothers, *Real Analysis,* Cambridge University Press, Cambridge (2000). Please read the relevant sections in the text as well as any cited references. Assignments are due the following Friday, or on April 30, whichever comes first.

Your written work reflects your professionalism. Please copy or paraphrase each question. Make answers complete and self contained, written in good technical English. This means that you should write in complete sentences, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer. Homework from Wednesday to Tuesday will be due Friday. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all. Homework that is placed in my mailbox in JWB 228 before 4 pm Friday afternoon will be considered to be on time.

Please hand in problems A1 – A3 on Friday, January 10.

A1. Equivalence. Exercises from Carothers' *Real Analysis.*

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20[8]
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A2. Is the set of all finite subsets of $\mathbb{N}$ countable or uncountable? Give a proof of your assertion. [Strichartz, p. 13]

A3. Let $A_1, A_2, A_3, \ldots$ be countably infinite sets, and let their Cartesian product $A_1 \times A_2 \times A_3 \times \cdots$ be defined as the set of all sequences $(a_1, a_2, a_3, \ldots)$ where a_k is an element of A_k, Prove that the Cartesian product is uncountable. Show that the same conclusion holds if each of the sets $A_1, A_2, \ldots$ has at least two elements. [Strichartz, p. 13]

Please hand in problems B1 on Friday, January 17.

B1. Cantor Set. Exercises from Carothers' *Real Analysis.*

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Chapter 2  Problems 18, 21, 26, 34.
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Please hand in problems C1–C4 on Friday, January 24.

C1. Associativity of Reals. Write out the proof of the associative law of addition for the real numbers. [Strichartz, p.48.]

C2. Increasing Representative. Let x be a real number. Show that there is a Cauchy sequence of rationals (b_n) representing x such that $b_n < b_{n+1}$ for every n. [Strichartz, p.49.]

C3. Density of Rationals. Prove that there are infinitely many rational numbers between any two distinct real numbers. [Haaser & Sullivan, p.34.]

C4. Order in Reals. Show that if a real number x can be represented by a Cauchy sequence of positive rationals, then $x \geq 0$. [Strichartz, p.49.]

Please hand in problems D1–D2 on Friday, January 31.

D1. Metric Spaces. Exercises from Carothers' *Real Analysis.*

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Chapter 3  Problems 6, 9, 20.
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D2. Inner Product Spaces. In an inner product space, prove the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \qquad \text{for all } x, y.$$

Prove that $\|x\|_1$ an $\mathbf{R}^n$ for $n > 1$ is not associated to an inner product. Do the same for $\|x\|_{\text{sup}}$. [Strichartz, p. 367.]

Please hand in problems E1 on Friday, February 7.

E1. Topology of Metric Spaces. Exercises from Carothers' *Real Analysis.*

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Chapter 3  Problem  44,
Chapter 4  Problem  4, 10, 11, 17, 20.
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Please hand in problems F1 on Friday, February 14.

F1. Continuity. Exercises from Carothers' *Real Analysis.*

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Chapter 5  Problem  17, 28, 37
Chapter 6  Problem  5
Chapter 7  Problem  10.
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Please hand in problems G1-G3 on Friday, February 21.

G1. Complete Metric Spaces. Exercises from Carothers' *Real Analysis.*

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Chapter 7  Problem  24, 32, 39.
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G2. Fredholm Integral Equation. Let $I = [a, b]$, $g : I \to \mathbf{R}^n$ and $K : I \times I \to M_{n\times n}(\mathbf{R})$ be continuous functions, where $M_{n\times n}(\mathbf{R})$ are real $n \times n$ matrices. Find $\lambda_0 > 0$ so that if $|\lambda| \le \lambda_0$, then there is a unique continuous function $x : I \to \mathbf{R}^n$ that solves the Fredholm Integral Equation. [Haaser & Sullivan, *Real Analysis,* p. 104.]

$$x(t) = \lambda \int_a^b K(t, s)x(s)\, ds + g(t).$$

G3. Completion. Let (M, d) be a metric space.

1. Call two Cauchy sequences (p_n) and (q_n) equivalent if
$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$
Prove this is an equivalence relation.
2. Let M^* be the equivalence classes of Cauchy Sequences in M. If $P^*, Q^* \in M^*$, $(p_n) \in P^*$, $(q_n) \in Q^*$ define
$$\Delta(P^*, Q^*) = \lim_{n\to\infty} d(p_n, q_n).$$
Show that this limit exists, is well defined and provides a distance function for M^*.
3. Prove that the resulting metric space (M^*, Δ) is complete.
4. For each $p \in M$ there is a Cauchy Sequence $(\bar{p})$ all of whose terms are p. Let $P_p = [(\bar{p})]$. prove that
$$\Delta(P_p, P_q) = d(p, q)$$
for all $p, q \in M$. In other words, the mapping $\varphi(p) = P_p$ is an isometry of M into M^*.
5. Prove that $\varphi(M)$ is dense on M^* and that $\varphi(M) = M^*$ if M is complete.
[Rudin, *Principles of Mathematical Analysis* 3rd ed., p. 82.]

Please hand in problems H1-H4 on Friday, February 28.

H1. Compact. Exercises from Carothers' *Real Analysis.*

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Chapter 8  Problem  13, 33, 56.
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H2. Hausdorff Metric. Find $d(A, B)$, $d(B, A)$ and $h(A, B)$, the Hausdorff distance for the given A and B.

a. $A = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 1\}$;
b. $A = \{(x, y) : |x| \le 1 \text{ and } |y| \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 1\}$;
c. $A = \{(x, y) : |x| \le 1 \text{ and } |y| \le 1\}$, $\quad B = \{(x, y) : x^2 + y^2 \le 2\}$;

H3. ϵ-Collars. Let $A, B, C, D \in \mathcal{K}(\mathbf{R}^n)$.

a. For $\epsilon > 0$ show $h(A_\epsilon, B_\epsilon) \le h(A, B)$, where $A_\epsilon = \{x \in \mathbf{R}^n : d(x, A) \le \epsilon\}$.

b. Let $A \boxplus B = \{a + b : a \in A,\ b \in B\}$ be the Minkowski sum. Show $h(A \boxplus B, C \boxplus D) \le h(A, C) + h(B, D)$.
[Hadwiger, *Vorlesungen Über Inhalt, Oberfläche und Isoperimetrie,* p. 152.]

H4. Decreasing Sequence. Let $K_n \in \mathcal{K}(\mathbf{R}^n)$ such that $K_n \supset K_{n+1}$ for all n. Show that in $(\mathcal{K}(\mathbf{R}^n), h)$,

$$\lim_{n\to\infty} K_n = K_\infty \qquad \text{where} \qquad K_\infty = \bigcap_{n=1}^{\infty} K_n.$$

[D. Burago, Y. Burago & S. Ivanov, *A Course in Metric Geometry.* p. 253.]

Please hand in problems I1 on Friday, March 7.

I1. Equivalent Metrics. Exercises from Carothers' *Real Analysis.*

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Chapter   8  Problems  79, 80, 86;
Chapter  10  Problems  11, 13.
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Please hand in problems J1 on Friday, March 21.

J1. Weierstrass Approximation. Exercises from Carothers' *Real Analysis.*

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Chapter  10  Problems  18, 31;
Chapter  11  Problems  7, 14, 38.
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Please hand in problems K1 on Friday, March 28.

K1. Arzela-Ascoli Theorem. Exercises from Carothers' *Real Analysis.*

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Chapter  11  Problems  43, 52, 63;
Chapter  12  Problems  23;
Chapter  16  Problems  9.
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Please hand in problems L1 on Friday, Apr. 4.

L1. Lebesgue Outer Measure. Exercises from Carothers' *Real Analysis.*

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Chapter  16  Problems  24, 25, 39, 41.
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Please hand in problems M1 on Friday, Apr. 11.

M1. Lebesgue Measurable Functions. Exercises from Carothers' *Real Analysis.*

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Chapter  16  Problems  48, 53, 73;
Chapter  17  Problems  3, 5.
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Please hand in problems N1 on Friday, Apr. 18.

N1. Lebesgue Integral. Exercises from Carothers' *Real Analysis.*

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Chapter  17  Problems  20, 26, 35;
Chapter  18  Problems  3, 6.
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All outstanding homework is due Apr. 18. No papers will be accepted after Apr. 29.

FINAL EXAM is Tue., Apr. 29 at 10:30 AM- 12:30 PM in the usual JWB 333.