Geodesics and the Gauß-Bonnet Theorem

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Abstract. In these notes we compute the geodesic curvature on a surface in isothermal coordinates and use it to prove the local Gauß-Bonnet Theorem.

These remarks are a continuation of my notes [T] whose notation we continue to employ.

1. Isothermal Coordinates of a Surface.

The computations are facilitated by using a special coordinate system in which the metric and the resulting formulas take a particularly simple form.

Theorem [Isothermal Coordinates]. *[Korn-Lichtenstein 1914, Lavrentiev, Morrey] Suppose M is a surface of class $C^{k,\alpha}$ ($k \in \mathbf{N}, 0 < \alpha < 1$) or C^∞ and $P \in M$. Then another coordinate patch $X : \Omega \to M$ with $P \in X(\Omega)$ may be found (Ω is an open set of $\mathbb{R}^2$) such that $X(u^1, u^2) \in C^{k,\alpha}(\Omega)$ or $C^\infty(\Omega)$, resp., such that the conformality relations $g_{11} = X_1 \cdot X_1 = g_{22} = X_2 \cdot X_2 > 0$ and $g_{12} = X_1 \cdot X_2 = 0$ are satisfied for all $(u^1, u^2) \in \Omega$.*

For a proof of this, see Jost [J]. Thus the metric $g_{11} = g_{22} = \varphi(u^1, u^2)$ is given by a single function. The Gauss Curvature formula for orthogonal coordinates reduces to

$$K = -\frac{1}{2\varphi}\Delta \log \varphi. \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial(u^1)^2} + \frac{\partial^2}{\partial(u^2)^2}$ is the usual Laplace operator.

We shall have occasion to use the Christoffel symbols associated to this metric.

$$\Gamma_{ij}{}^m = \frac{1}{2} g^{\ell m}\left\{\frac{\partial}{\partial u^j} g_{i\ell} + \frac{\partial}{\partial u^i} g_{\ell j} - \frac{\partial}{\partial u^\ell} g_{ji}\right\}. \tag{2}$$

Let us assume that $g_{12} = 0$ and $g_{11} = g_{22} = \varphi$ on all of our coordinate patch Ω. Thus also $g^{ij} = \delta^{ij}/\varphi$. Then using (2), the Christoffel symbols take the form

$$\begin{aligned} \Gamma_{11}{}^1 &= \frac{\varphi_1}{2\varphi} & \Gamma_{11}{}^2 &= -\frac{\varphi_2}{2\varphi} \\ \Gamma_{21}{}^1 = \Gamma_{12}{}^1 &= \frac{\varphi_2}{2\varphi} & \Gamma_{21}{}^2 = \Gamma_{12}{}^2 &= \frac{\varphi_1}{2\varphi} \\ \Gamma_{22}{}^1 &= -\frac{\varphi_1}{2\varphi} & \Gamma_{22}{}^2 &= \frac{\varphi_2}{2\varphi}. \end{aligned} \tag{3}$$

2. Geodesic Curvature of a curve on the surface a Surface.

Consider a curve $\gamma(s) \in M$ in the patch parameterized by arclength. Then, the tangent vector $T = \gamma'$ is a unit vector tangent to M where "$'$" denotes $\frac{d}{ds}$. The surface normal vector U is another unit vector orthogonal to T. Following the text, we denote the third vector $W = U \times T$ so that $\{T, W, U\}$ is a moving orthonormal (right handed) frame along γ adapted to the surface. The normal curvature and the geodesic curvature are the normal and tangent components of T'. We considered such a frame when we discussed White's formua relating the linking number to the twisting number of a pair of closed curves [O, p. 24]. Starting with orthonormality $T \cdot T = W \cdot W = U \cdot U = 1$ and $T \cdot W = T \cdot U = W \cdot U = 0$, by differentiating we find generalized Frenet equations for the accelerations

$$\begin{cases} T' = & + \omega_3 W - \omega_2 U \\ W' = - \omega_3 T & + \omega_1 U \\ U' = \omega_2 T - \omega_1 W & \end{cases}$$

where the ω_i are functions. If we regard $\gamma \in M \subset \mathbb{R}^3$ as a space curve, then we may suppose that it has the usual orthonormal Frenet Frame $\{T, N, B\}$ moving with the curve. Since N is a unit vector perpendicular to T, we may write

$$N = \cos(\theta)\, W + \sin(\theta)\, U \tag{4}$$

where the function $\theta = \angle(N, W)$ gives the angle between M and W. The Frenet equation defines the curvature κ and the torsion τ of the space curve by the formulæ

$$\begin{cases} T' = & + \kappa N & \\ N' = - \kappa T & & + \tau B \\ B' = & - \tau N & \end{cases}$$

The normal curvature is given in terms of the shape operator, thus differentiating $U \cdot T = 0$, and using $T' = \kappa N$,

$$k_n = \mathcal{S}(T) \cdot T = -U' \cdot T = U \cdot T' = U \cdot \kappa N = \kappa \sin(\theta) = -\omega_2.$$

Let us define the *geodesic curvature* k_g as the other component

$$k_g = W \cdot T' = W \cdot \kappa N = \kappa \cos(\theta) = \omega_3.$$

Thus the normal and geodesic curvatures satisfy the relation

$${k_n}^2 + {k_g}^2 = \kappa^2.$$

It remains to determine ω_1. Using the right-handedness of the frame, $T \times W = U$, $W \times U = T$ and $U \times T = W$,

$$B = T \times N = \cos(\theta)\, T \times W + \sin(\theta)\, T \times U = -\sin(\theta)\, W + \cos(\theta)\, U. \tag{5}$$

Inverting the relations (4) and (5) we find

$$\begin{aligned} W &= \cos(\theta)\, N - \sin(\theta)\, B \\ U &= \sin(\theta)\, N + \cos(\theta)\, B. \end{aligned}$$

Differentiating using the Frenet equation yields

$$\begin{aligned} W' &= \cos(\theta)\, N' - \sin(\theta)\, B' + (-\sin(\theta)\, N - \cos(\theta)\, B)\, \theta' \\ &= \cos(\theta)\, (-\kappa\, T + \tau\, B) + \tau \sin(\theta)\, N - \theta'\, U \\ &= -\, k_g\, T + (\tau - \theta')\, U \end{aligned}$$

whence $\omega_1 = U \cdot W' = \tau - \theta'$.

The generalized Frenet Equations become

$$\begin{cases} T' = & + k_g\, W & + \quad k_n\, U \\ W' = -\, k_g\, T & & + (\tau - \theta')\, U \\ U' = -\, k_n\, T - (\tau - \theta')\, W & & \end{cases}$$

The curve γ is called a *geodesic,* if it is as straight as possible on the surface, which means that there is no tangential acceleration, $k_g = 0$. Let us show that the only geodesics on the sphere $\mathbb{S}^2_r$ are great circles. Consider a sphere of radius r centered at the origin, and tahe the outward unit normal

$$\frac{1}{r} X = U.$$

A curve $\alpha(s)$ on $\mathbb{S}^2_r$ parameterized by arclength satisfies $\alpha \cdot \alpha = r^2$. Differentiating, $\alpha \cdot T = 0$ and $T \cdot T + \kappa \alpha \cdot N = 0$ which implies that the curvature $\kappa > 0$ is nonzero. Since we are assuming $k_g = 0$ for α, its acceleration is purely in the normal direction, or

$$\kappa N = T' = k_n U = \frac{k_n}{r} X$$

Thus $\kappa \neq 0$ implies $N = fX$ for some function f. Thus since the lengths are known, $f = \pm\frac{1}{r}$ which is constant along α. Differentiating again using both Frenet systems,

$$-\kappa T + \tau B = N' = fT$$

from which it follows that $\tau = 0$ and $\kappa = \frac{1}{r}$. However a space curve with $\tau = 0$ is planar and with $\kappa = \frac{1}{r}$ is a circle of radius r. Thus α is a great circle of $\mathbb{S}^2_r$.

The tangential part of the motion of the curve is intrinsic to the surface, thus it may be computed strictly from the curve and the first fundamental form and its derivatives. Recall the acceleration formulæ in arbitrary coordinates.

$$\begin{aligned} U_i &= \ - h_{ij}\, g^{jk}\, X_k, \\ X_{ij} &= \Gamma_{ij}{}^k\, X_k + h_{ij} U, \end{aligned}$$

A curve on the surface may be given by

$$\gamma(s) = X(u^1(s), u^2(s))$$

Its velocity and acceleration is thus by the chain rule and acceleration formulæ

$$\begin{aligned} T =& \gamma' = X_i \dot{u}^i, \\ T' =& X_i \ddot{u}^i + X_{ij} \dot{u}^i\, \dot{u}^j = \left(\ddot{u}^k + \Gamma_{ij}{}^k \dot{u}^i\, \dot{u}^j \right)\, X_k + \left(h_{ij}\, \dot{u}^i\, \dot{u}^j \right)\, U \end{aligned} \tag{6}$$

where now "˙" denotes $\frac{d}{ds}$. In our notation, the geodesic curvature and the normal curvature are given by

$$\begin{aligned} k_n =& T' \cdot U = h_{ij}\, \dot{u}^i\, \dot{u}^j \\ k_g =& T' \cdot W = \left(\ddot{u}^k + \Gamma_{ij}{}^k \dot{u}^i\, \dot{u}^j \right)\, X_k \cdot W \end{aligned} \tag{7}$$

It follows that if the curve γ is a geodesic, $k_g = 0$, and since $T' \cdot T = 0$, then the equations of a geodesic curve are given by

$$\ddot{u}^k(s) + \Gamma_{ij}{}^k[u^1(s), u^2(s)]\, \dot{u}^i(s)\, \dot{u}^j(s) = 0 \qquad \text{for } k = 1, 2. \tag{8}$$

This is a second order nonlinear 2×2 system of ordinary differential equations on the surface, involving only the first fundamental form.

Suppose that $\gamma(\sigma) = X(u^1(\sigma), u^2(\sigma))$ is a curve which is not necessarily given with arclength parameter. Then $|X_i\dot u^i|^2 = g_{ij}\,\dot u^i\,\dot u^j$ so

$$\begin{aligned}
T =& \frac{\gamma_\sigma}{|\gamma_\sigma|} = \frac{X_i\dot u^i}{|X_i\dot u^i|} = \frac{X_i\dot u^i}{|X_i\dot u^i|},\\
\frac{d\sigma}{ds} =& \frac{1}{\sqrt{g_{ij}\,\dot u^i\,\dot u^j}}\\
T' =& \dot T\frac{d\sigma}{ds} = \frac{X_i\ddot u^i + X_{ij}\dot u^i\,\dot u^j}{\sqrt{g_{ij}\,\dot u^i\,\dot u^j}} - \frac{\frac{d}{d\sigma}\left[g_{ij}\,\dot u^i\,\dot u^j\right]T}{2g_{ij}\,\dot u^i\,\dot u^j}\\
=& \frac{\left(\ddot u^k + {\Gamma_{ij}}^k\dot u^i\,\dot u^j\right)\,X_k + \left(h_{ij}\,\dot u^i\,\dot u^j\right)\,U}{\sqrt{g_{ij}\,\dot u^i\,\dot u^j}} - \left(\frac{d}{d\sigma}\log\sqrt{g_{ij}\,\dot u^i\,\dot u^j}\right)T
\end{aligned}$$

where in this equation, "$\cdot$" denotes $\frac{d}{d\sigma}$. Since (7), the equations for a geodesic continue to be (8), even if s is not arclength and g_{ij} is any metric.

For example, in the Euclidean plane, the metric $g_{ij} = \delta_{ij}$ are constant functions so that ${\Gamma_{ij}}^k = 0$ and the geodesic equations reduce to $\ddot u^i = 0$. In other words $u^i = a^i s + b^i$ for constants a^i, b^i, which are linear functions. Thus the geodesics are straight lines.

In principle, one can work out W in terms of $X_i\dot u^i$ and g_{ij} for to get an intrinsic expression for k_g for any metric. But it is easier to work in a nice coordinate system, thus now we assume that $X(u^1, u^2)$ is an isothermal coordinate patch. In this case the X_i are orthogonal and since $|X_i| = \sqrt{\varphi}$, we have a new orthonormal basis given by

$$\mathbf{e}_i = \frac{1}{\sqrt{\varphi}}X_i.$$

Thus we can write the two orthonormal tangent vectors of (6),

$$T = \sqrt{\varphi}\,\dot u^1\mathbf{e}_1 + \sqrt{\varphi}\,\dot u^2\mathbf{e}_2, \qquad W = -\sqrt{\varphi}\,\dot u^2\mathbf{e}_1 + \sqrt{\varphi}\,\dot u^1\mathbf{e}_2 = -\dot u^2\,X_1 + \dot u^1\,X_2$$

Substituting into (7), using $X_i\cdot X_j = \varphi\,\delta_{ij}$,

$$k_g = \left(\ddot u^k + {\Gamma_{ij}}^k\dot u^i\,\dot u^j\right)X_k\cdot\left(-\dot u^2\,X_1 + \dot u^1\,X_2\right) = -\varphi\left(\ddot u^1 + {\Gamma_{ij}}^1\dot u^i\,\dot u^j\right)\,\dot u^2 + \varphi\left(\ddot u^2 + {\Gamma_{ij}}^2\dot u^i\,\dot u^j\right)\dot u^1$$

Finally, we substitute the expressions (3) for the Christoffel symbols,

$$\begin{aligned}
\dot u^1{\Gamma_{11}}^2 - \dot u^2{\Gamma_{11}}^1 =& -\frac{\dot u^1\,\varphi_2}{2\varphi} - \frac{\dot u^2\,\varphi_1}{2\varphi}\\
\dot u^1{\Gamma_{12}}^2 - \dot u^2{\Gamma_{12}}^1 =& \quad\frac{\dot u^1\,\varphi_1}{2\varphi} - \frac{\dot u^2\,\varphi_2}{2\varphi}\\
\dot u^1{\Gamma_{22}}^2 - \dot u^2{\Gamma_{22}}^1 =& \quad\frac{\dot u^1\,\varphi_2}{2\varphi} + \frac{\dot u^2\,\varphi_1}{2\varphi}
\end{aligned}$$

and find using the unit speed $|\gamma'| = 1$ so $(\dot u^1)^2 + (\dot u^2)^2 = \frac{1}{\varphi}$

$$\begin{aligned}
k_g =& \varphi\left(\ddot u^2\,\dot u^1 - \ddot u^1\,\dot u^2\right) + \varphi\left(\dot u^1{\Gamma_{ij}}^2 - \dot u^2{\Gamma_{ij}}^1\right)\,\dot u^i\,\dot u^j\\
=& \varphi(\ddot u^2\,\dot u^1 - \ddot u^1\,\dot u^2) - \tfrac12\left(\dot u^1\varphi_2 + \dot u^2\varphi_1\right)\,(\dot u^1)^2 + \left(\dot u^1\varphi_1 - \dot u^2\varphi_2\right)\,\dot u^1\,\dot u^2 + \tfrac12\left(\dot u^1\varphi_2 + \dot u^2\varphi_1\right)\,(\dot u^2)^2\\
=& \varphi\left(\ddot u^2\,\dot u^1 - \ddot u^1\,\dot u^2\right) - \tfrac12\left(\varphi_2\,(\dot u^1)^3 - \varphi_1\,(\dot u^1)^2\,\dot u^2 + \varphi_2\,\dot u^1\,(\dot u^2)^2 - \varphi_1\,(\dot u^2)^3\right)\\
=& \varphi(\ddot u^2\,\dot u^1 - \ddot u^1\,\dot u^2) + \tfrac12\left[\varphi_1\,\dot u^2 - \varphi_2\,\dot u^1\right]\left((\dot u^1)^2 + (\dot u^2)^2\right)\\
=& \frac{\ddot u^2\,\dot u^1 - \ddot u^1\,\dot u^2}{(\dot u^1)^2 + (\dot u^2)^2} + \frac{\varphi_1\,\dot u^2 - \varphi_2\,\dot u^1}{2\varphi}
\end{aligned}$$

Finally, if we denote the angle that the curve makes relative to the (u^1, u^2) coordinates by η, that is $(\cos\eta, \sin\eta) = \sqrt{\varphi}\,(\dot u^1, \dot u^2)$, we find

$$k_g = \eta' + \tfrac12\left[(\log\varphi)_1\,\dot u^2 - (\log\varphi)_2\,\dot u^1\right] \tag{8}$$

where we have differentiated $\eta = \tan^{-1}\left(\dfrac{\dot u^2}{\dot u^1}\right)$. Oprea [O, p. 308] deduces the same formula for an orthogonal coordinate system.

3. Gauß Bonnet Formula closed curves in a coordinate patch.

Suppose γ is a closed simple curve in an isothermal coordinate patch. Assume that all of the interior region bounded by the curve is included in the patch such that the curve can be continuously contracted to a point through curves in the patch. We shall call such curves *simply connected.* A picewise smooth curve may have finitely many kinks. At each kink $\gamma(s_i)$, where $s_1 < s_2 < \cdots < s_n$ are the nonsmooth points along the curve, let α_i denote the exterior angle. That is $\alpha_i = \angle(\gamma'(s_i + 0), \gamma'(s_i - 0))$ is the angle from the ending tangent direction of the smooth curve before s_i on $[s_{i-1}, s_i]$ to the starting tangent direction of the smooth curve after s_i on $[s_i, s_{i+1}]$. One of the advantages of the coordinates being isothermic is that the angle between two vectors in the (u^1, u^2) plane is the same as the angle between corresponding vectors on the surface M. Thus $\alpha_i = \eta(s_i + 0) - \eta(s_i - 0)$ is also the jump in the angle along the curves in the (u^1, u^2) plane at s_i.

Theorem. *Let γ be a piecewise smooth, regular, simple, simply connected curve in an isothermal coordinate patch. Let D be the region inside the patch whose boundary is $\gamma = dD$. Then*

$$\iint_D K\,dA + \oint_\gamma k_g\,ds + \sum_{i=1}^n \alpha_i = 2\pi.$$

Proof. We apply the divergence theorem. Use the fact that the area form on the surface is given by $dA = \sqrt{g_{11}g_{22} - {g_{12}}^2}\,du^1\,du^2 = \varphi\,du^1\,du^2$. Also, the outside unit normal vector for a curve $(u^1(s), u^2(s))$ going around D in the counterclockwise direction is given by $\nu = \sqrt{\varphi}(\dot u^2, -\dot u^1)$ and the arclength in the (u^1, u^2) plane is $ds/\sqrt{\varphi}$. Thus we find from (1) and (8) that

$$\begin{aligned}\iint_D K\,dA &= -\iint_D \left(\frac{\Delta \log\varphi}{2\varphi}\right)\varphi\,du^1\,du^2 = -\frac{1}{2}\oint_\gamma \nabla\log\varphi\cdot\nu\,\frac{ds}{\sqrt{\varphi}}\\ &= -\frac{1}{2}\oint_\gamma \dot u^2(\log\varphi)_1 - \dot u^1(\log\varphi)_2\,ds = \oint \eta' - k_g\,ds\\ &= -\oint k_g\,ds + 2\pi - \sum_{i=1}^n \alpha_i.\end{aligned}$$

The last inequality follows from fact that the total angle $\eta(L) - \eta(0)$ turned by a simple, simply connected curve is 2π. Here L is the length of γ. The contribution to the total turning at each corner is given by α_i. One can imagine approximating the curve near a corner by a very tight, yet smooth curve that is rounded at that corner, and which turns an angle α_i near the corner. The fact that the total turn of a piecewise regular simple, simply connected closed curve in the (u^1, u^2) plane is 2π is known as the *Umlaufsatz*, [dC, p. 396.] The integral of geodesic curvature plus the total of the exterior angles is the total turning 2π. □

By covering a compact surface with piecewise smooth regions from isothermal charts, the global Gauß-Bonnet formula may be proved as in the text.

References

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