Gauß Curvature in Terms of the First Fundamental Form

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Abstract. In these notes, we develop acceleration formulae for a general frame for a surface in three space. We prove Gauß's Theorem and compute some examples for a general metric. This is a supplement to the Oprea's text [O] which only treats the special case that the cross terms vanish. We show how that case follows from the general one, and derive the simplified expression for the Gauß curvature.

We first recall the definitions of the first and second fundamental forms of a surface in three space. We develop some tensor notation, which will serve to shorten the expressions. We then compute the Gauß and Weingarten equations for the surface. We find the Codazzi Equation and show Gauß's Theorema Egregium. Finally we work some examples and write the simplified expression in lines of curvature coordinates.

1. First and Second Fundamental Forms of a Surface.

Met $M \subset \mathbb{R}^3$ denote a smooth regular surface. We suppose that a local parameterization for M be given by a cordinate patch $X(u,v) : \Omega \to M$ where $\Omega \subset \mathbb{R}^2$ is an open domain. Tangent vectors are the partial derivatives of the position function, giving vector functions X_u and X_v which are independent for all $(u,v) \in \Omega$ by the assumption of regularity. The normal vector field on the surface is given locally by the vector function

$$U(u,v) = \frac{X_u \times X_v}{|X_u \times X_v|}.$$

For simplicity we shall denote $u^1 = u$, $u^2 = v$, and differentiation $f_i = \dfrac{\partial f}{\partial\, u^i}$ so that

$$X_1 = \frac{\partial}{\partial\, u^1} X = X_u, \qquad X_2 = \frac{\partial}{\partial u^2} X = X_v.$$

The shape operator is the linear transformation on tangent vectors $W \in T_pM$ given by the formula

$$S(W) = -\nabla_W U.$$

A general vector can be written in terms of the basis, $W = w^1 X_1 + w^2 X_2$ where $w^i(u^1, u^2)$, $i = 1, 2$, are functions on Ω. By linearity,

$$S(W) = S(w^1 X_1 + w^2 X_2) = w^1 S(X_1) + w^2 S(X_2). \tag{1}$$

Using the fact that the normal vector field has unit length, $U \cdot U = 1$, we find by differentiating, $U_i \cdot U = 0$ so that

$$S(X_i) = -\nabla_{X_i} U = -\frac{\partial}{\partial u^i} U = -U_i \tag{2}$$

is perpendicular to U, thus in T_pM. Hence by (1), $S(W) \in T_pM$ is a tangent vector. Similarly, since tangent and normal vectors are perpendicular, $U \cdot X_j = 0$ implies by differentiating

$$U_i \cdot X_j + U \cdot X_{ij} = 0 \tag{3}$$

The first and second fundamental forms are the matrix functions, which by (2) and (3) are

$$g_{ij} = X_i \cdot X_j, \qquad h_{ij} = S(X_i) \cdot X_j = -U_i \cdot X_j = U \cdot X_{ij}. \tag{4}$$

By the independence of X_1, X_2, we see that g_{ij} is a positive definite matrix function. Also (4) implies that both matrix functions are symmetric $g_{ij} = g_{ji}$ and $h_{ij} = h_{ji}$ because cross partial derivatives are equal $X_{ij} = X_{ji}$. Thus also $g^{ij} = g^{ji}$.

2. Some Tensor Notation.

For compactifying the formulas, it is convenient to introduce some manipulations of functions, vectors, matricies and other multiindex functions, which are all *tensors.* The indices are assumed to take the values $i, j, k, \ldots \in \{1, 2\}$. The components of the identity matrix function I are given by the Kronecker-delta

$$\delta_i{}^j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}, \qquad I = \begin{pmatrix} \delta_1{}^1 & \delta_1{}^2 \\ \delta_2{}^1 & \delta_2{}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Kronecker delta may appear with indices up or down: $\delta_{ij} = \delta^i{}_j = \delta^{ij} = \delta_i{}^j$. If we denote the matrix whose entries are g_{ij} by $\mathcal{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, then it is convenient to denote the determinant by $g = g_{11}g_{22} - {g_{12}}^2$ and the entries of $\mathcal{G}^{-1}$ by g^{ij}. That is

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{g} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}.$$

Let us also denote the matrix whose entries are h_{ij} by $\mathcal{H}$.

We shall assume the *Einstein Summation Convention*, which is that whenever an index is repeated in a formula, the formula is to be summed over that index. Thus, the expression $h_{ij}g^{jk}$ really means

$$h_{ij}g^{jk} = \sum_{j=1}^{2} h_{ij}g^{jk}.$$

This is just matrix multiplication. Thus the (i, k) components of the matrix equations $\mathcal{G}\mathcal{G}^{-1} = I$ and $\mathcal{H}I = \mathcal{H}$ are

$$g_{ij}g^{jk} = \delta_i{}^k, \qquad h_{ij}\delta^j{}_k = h_{ik}.$$

Finally, this enables formulas like

$$h_{ij}g^{jk}g_{k\ell} = h_{ij}\delta^j{}_\ell = h_{i\ell}.$$

3. Acceleration Formulæ and Christoffel Symbols.

It is useful to compute the expressions of the change of the local vector field basis $\{X_1, X_2, U\}$ as points change along the surface. This is the surface analog of the Frenet Equations for the moving frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along a space curve which expresses derivatives of the frame vectors in terms of the vectors and the curvature and torsion. Since $U \cdot U = 1$ implies that its derivatives $U_i \cdot U = 0$ so U_i is tangent to the surface. Thus the acceleration vectors may be decomposed in terms of the basis as

$$\begin{aligned} U_i &= -\, a_i{}^j \, X_j, \\ X_{ij} &= \Gamma_{ij}{}^k \, X_k + b_{ij}U, \end{aligned} \tag{5}$$

where the $a_i{}^j$, b_{ij} and $\Gamma_{ij}{}^k$ are unknown coefficient functions defined on Ω. The first of these equations is called the *Gauß Equation.* The second is called the *Weingarten Equation.* We have already seen the Gauß Equation. The matrix of the shape operator is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which means by (2),

$$\begin{aligned} S(X_1) &= -U_1 &&= aX_1 + bX_2 \\ S(X_2) &= -U_2 &&= cX_1 + dX_2 \end{aligned}$$

By (5.1) this means

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} {a_1}^1 & {a_1}^2 \\ {a_2}^1 & {a_2}^2 \end{pmatrix}.$$

Let us rederive the formula for ${a_i}^j$ as was done in the text. Taking an inner product of (5.1) with X_k, we find using (4),

$$-h_{ik} = U_i \cdot X_k = -{a_i}^j X_j \cdot X_k = -{a_i}^j g_{jk}.$$

Postmultiplying by $-g^{k\ell}$ gives the desired formula

$$\tag{6} {a_i}^\ell = {a_i}^j {\delta_j}^\ell = {a_i}^j g_{jk}\, g^{k\ell} = h_{ik}\, g^{k\ell}.$$

The principal curvatures k_i are the eigenvalues of the shape operator, which are functions on Ω. At each point there are two independent unit eigenvector functions V_i such that $S(V_i) = k_i V_i$ on Ω. The, Gauß curvature K and the mean curvature H are the determinant and trace of the shape operator. In terms of its matrix $({a_i}^j)$ in the $\{X_1, X_2\}$ basis these have the expressions

$$K = k_1 k_2 = \det({a_i}^k) = \det(h_{ij} g^{jk}) = \frac{\det(h_{ij})}{\det(g_{ij})},$$
$$H = \tfrac{1}{2}(k_1 + k_2) = \tfrac{1}{2}\operatorname{tr}({a_i}^j) = \tfrac{1}{2}\operatorname{tr}(h_{ij} g^{jk}) = \tfrac{1}{2} h_{ij} g^{ij}.$$

Using (4), $U \cdot U = 1$, $U \cdot X_k = 0$ and taking the inner product (5.2) with U yields the second formula

$$\tag{7} h_{ij} = X_{ij} \cdot U = {\Gamma_{ij}}^k\, X_k \cdot U + b_{ij} U \cdot U = b_{ij}$$

Finding the Christoffel Symbols requires a little trick. Taking inner products of (5.2) with X_ℓ we find

$$\tag{8} X_{ij} \cdot X_\ell \;=\; {\Gamma_{ij}}^k\, X_k \cdot X_\ell + b_{ij} U \cdot X_\ell \;=\; {\Gamma_{ij}}^k\, g_{k\ell}.$$

It follows that ${\Gamma_{ij}}^m = X_{ij} \cdot X_k\, g^{km}$ so ${\Gamma_{ij}}^m = {\Gamma_{ji}}^m$ for all i, j, m. To find the left hand side, we observe that the derivative gives desired term plus a garbage term

$$\frac{\partial}{\partial u^j} g_{i\ell} \;=\; \frac{\partial}{\partial u^j}\left(X_i \cdot X_\ell\right) \;=\; X_{ij} \cdot X_\ell + X_i \cdot X_{\ell j}.$$

To cancel off the garbage term we add three such derivatives in which the indices are cyclically permuted

$$\begin{array}{rl}
\dfrac{\partial}{\partial u^j} g_{i\ell} & = X_{ij} \cdot X_\ell + X_i \cdot X_{\ell j} \\
\dfrac{\partial}{\partial u^i} g_{\ell j} & = X_{\ell i} \cdot X_j + X_\ell \cdot X_{ji} \\
-\dfrac{\partial}{\partial u^\ell} g_{ji} & = -X_{j\ell} \cdot X_i - X_j \cdot X_{i\ell} \\
\hline
\dfrac{\partial}{\partial u^j} g_{i\ell} + \dfrac{\partial}{\partial u^i} g_{\ell j} - \dfrac{\partial}{\partial u^\ell} g_{ji} & = 2X_{ij} \cdot X_\ell
\end{array}$$

Postmultiplying (8) by $g^{\ell m}$ yields

$$\tag{9} {\Gamma_{ij}}^m = {\Gamma_{ij}}^k g_{k\ell} g^{\ell m} = \frac{1}{2} g^{\ell m} \left\{ \frac{\partial}{\partial u^j} g_{i\ell} + \frac{\partial}{\partial u^i} g_{\ell j} - \frac{\partial}{\partial u^\ell} g_{ji} \right\}.$$

(5), (6), (7) and (9) are summarized as follows.

Acceleration Formulæ. *The frame $\{X_1, X_2, U\}$ of a surface changes according to*

$$U_i = \, - h_{ij}\, g^{jk}\, X_k, \qquad X_{ij} = \Gamma_{ij}{}^{k}\, X_k + h_{ij} U, \tag{10}$$

where the Christoffel symbols are given by equation(9).

4. The Codazzi Equation and Gauß's Theorem.

It all follows from $0 = X_{ijk} - X_{ikj}$. Differentiating (5.2) we find using (5) and renaming a dummy index

$$\begin{aligned}
0 =& \frac{\partial}{\partial u^k} X_{ij} - \frac{\partial}{\partial u^j} X_{ik} \\
=& \frac{\partial}{\partial u^k} \left(\Gamma_{ij}{}^{\ell}\, X_\ell + h_{ij} U\right) - \frac{\partial}{\partial u^j} \left(\Gamma_{ik}{}^{\ell}\, X_\ell + h_{ik} U\right) \\
=& \left(\frac{\partial}{\partial u^k}\Gamma_{ij}{}^{\ell} - \frac{\partial}{\partial u^j}\Gamma_{ik}{}^{\ell}\right) X_\ell + \Gamma_{ij}{}^{\ell}\, X_{\ell k} - \Gamma_{ik}{}^{\ell}\, X_{\ell j} + \left(\frac{\partial}{\partial u^k} h_{ij} - \frac{\partial}{\partial u^j} h_{ik}\right) U + h_{ij} U_k - h_{ik} U_j \\
=& \left(\frac{\partial}{\partial u^k}\Gamma_{ij}{}^{m} - \frac{\partial}{\partial u^j}\Gamma_{ik}{}^{m}\right) X_m + \Gamma_{ij}{}^{\ell}\, \left(\Gamma_{\ell k}{}^{m}\, X_m + h_{\ell k} U\right) - \Gamma_{ik}{}^{\ell}\, \left(\Gamma_{\ell j}{}^{m}\, X_m + h_{\ell j} U\right) \\
&+ \left(\frac{\partial}{\partial u^k} h_{ij} - \frac{\partial}{\partial u^j} h_{ik}\right) U - \left(h_{ij} h_{k\ell} g^{\ell m} - h_{ik} h_{j\ell} g^{\ell m}\right) X_m.
\end{aligned}$$

Collecting the coefficients of the U basis vector yields the *Codazzi Equation*

$$\frac{\partial}{\partial u^k} h_{ij} - \frac{\partial}{\partial u^j} h_{ik} = \Gamma_{ik}{}^{\ell}\, h_{\ell j} - \Gamma_{ij}{}^{\ell}\, h_{\ell k}.$$

Collecting the coefficients of the X_m basis vector yields

$$0 = \left(\frac{\partial}{\partial u^k}\Gamma_{ij}{}^{m} - \frac{\partial}{\partial u^j}\Gamma_{ik}{}^{m}\right) + \Gamma_{ij}{}^{\ell}\, \Gamma_{\ell k}{}^{m} - \Gamma_{ik}{}^{\ell}\, \Gamma_{\ell j}{}^{m} - \left(h_{ij} h_{k\ell} - h_{ik} h_{j\ell}\right) g^{\ell m}.$$

Postmultiplying by g_{mp} and rearranging gives

$$h_{ij} h_{kp} - h_{ik} h_{jp} = \left(h_{ij} h_{k\ell} - h_{ik} h_{j\ell}\right) g^{\ell m}\, g_{mp} = \left(\frac{\partial}{\partial u^k}\Gamma_{ij}{}^{m} - \frac{\partial}{\partial u^j}\Gamma_{ik}{}^{m} + \Gamma_{ij}{}^{\ell}\, \Gamma_{\ell k}{}^{m} - \Gamma_{ik}{}^{\ell}\, \Gamma_{\ell j}{}^{m}\right) g_{mp}.$$

Finally, we can use this formula with $i = j = 1$, $k = p = 2$ to deduce a formula for the Gauss Curvature

$$K = \frac{h_{11} h_{22} - h_{12}{}^2}{g_{11} g_{22} - g_{12}{}^2} = \frac{1}{g} \left(\frac{\partial}{\partial u^2}\Gamma_{11}{}^{m} - \frac{\partial}{\partial u^1}\Gamma_{12}{}^{m} + \Gamma_{11}{}^{\ell}\, \Gamma_{\ell 2}{}^{m} - \Gamma_{12}{}^{\ell}\, \Gamma_{\ell 1}{}^{m}\right) g_{m2}. \tag{11}$$

This expression depends only on g_{ij} and its first and second derivatives. Thus we have proved Gauß's Gregarious (Egregious?) Theorem. (The Latin word *egregius* means "separated from the herd.")

Theorema Egregium. [Gauß, 1827] *The Gauß Curvature of a surface K is an intrinsic quantity. It depends only on the first fundamental form g_{ij} and its first and second derivatives. K is given by* (11)*, where the Christoffel symbols are computed using* (9).

5. Example: the curvature of the weaver's metric.

The computation of the curvature just in terms of g_{ij} is done systematically, by listing all of the Christoffel symbols in turn using (9), and then using (11). We shall do another computation in case then metric has $g_{12} = 0$ on all of Ω in Section 6.

An abstract piece of woven cloth can be thought of vertical and horizontal threads (the warp and the woof.) If the cloth is to be draped over a surface without tearing or bunching up, then the little squares of the fabric can distort to little rhombuses with angle $\varphi(u^1, u^2) \in (0, \pi)$. The fabric stretches along the bias but not along the threads. The mathematical way to say this is to require that the mapping $X : \Omega \to M$, the laying on of the cloth, does not stretch distances along u^i, but allows angular distortion. So the weaver's metric satisfies the equation

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & \cos\varphi \\ \cos\varphi & 1 \end{pmatrix}.$$

A coordinate system in which the metric takes this form is also called *Chebyshev coordinates.* This matrix is positive definite and $g = 1 - \cos^2\varphi = \sin^2\varphi$. Its inverse is given by

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \csc^2\varphi & -\cot\varphi\csc\varphi \\ -\cot\varphi\csc\varphi & \csc^2\varphi \end{pmatrix}.$$

Writing $g_{ij,k} = \frac{\partial}{\partial u^k} g_{ij}$, we see that $g_{11,i} = g_{22,i} = 0$ and $g_{12,i} = -\sin(\varphi)\varphi_i$. Hence the Christoffel symbols are computed according to (9)

$$\begin{aligned}
\Gamma_{11}{}^1 &= \tfrac12 g^{1i}\left(2g_{1i,1} - g_{11,i}\right) = \tfrac12 g^{11} g_{11,1} + g^{12}\left(g_{12,1} - \tfrac12 g_{11,2}\right) = \cot(\varphi)\varphi_1\\
\Gamma_{21}{}^1 = \Gamma_{12}{}^1 &= \tfrac12 g^{1i}\left(g_{1i,2} + g_{2i,1} - g_{12,i}\right) = \tfrac12 g^{11}\, g_{11,2} + \tfrac12 g^{12}\, g_{22,1} = 0\\
\Gamma_{22}{}^1 &= \tfrac12 g^{1i}\left(2g_{2i,2} - g_{22,i}\right) = g^{11}\left(g_{21,2} - \tfrac12 g_{22,1}\right) + \tfrac12 g^{12}\, g_{22,2} = -\csc(\varphi)\varphi_2\\
\Gamma_{11}{}^2 &= \tfrac12 g^{2i}\left(2g_{1i,1} - g_{11,i}\right) = \tfrac12 g^{21} g_{11,1} + g^{22}\left(g_{12,1} - \tfrac12 g_{11,2}\right) = -\csc(\varphi)\varphi_1\\
\Gamma_{21}{}^2 = \Gamma_{12}{}^2 &= \tfrac12 g^{2i}\left(g_{1i,2} + g_{2i,1} - g_{12,i}\right) = \tfrac12 g^{21}\, g_{11,2} + \tfrac12 g^{22}\, g_{22,1} = 0\\
\Gamma_{22}{}^2 &= \tfrac12 g^{2i}\left(2g_{2i,2} - g_{22,i}\right) = g^{21}\left(g_{21,2} - \tfrac12 g_{22,1}\right) + \tfrac12 g^{22}\, g_{22,2} = \cot(\varphi)\varphi_2
\end{aligned}\tag{12}$$

Substituting into (10) using $\Gamma_{12}{}^k = 0$ yields

$$\begin{aligned}
K &= \frac1g\left(\frac{\partial}{\partial u^2}\Gamma_{11}{}^m - \frac{\partial}{\partial u^1}\Gamma_{12}{}^m + \Gamma_{11}{}^\ell\,\Gamma_{\ell 2}{}^m - \Gamma_{12}{}^\ell\,\Gamma_{\ell 1}{}^m\right) g_{m2}\\
&= \frac{g_{12}}{g}\left(\frac{\partial}{\partial u^2}\Gamma_{11}{}^1 + \Gamma_{11}{}^2\,\Gamma_{22}{}^1\right) + \frac{g_{22}}{g}\left(\frac{\partial}{\partial u^2}\Gamma_{11}{}^2 + \Gamma_{11}{}^2\,\Gamma_{22}{}^2\right)\\
&= \cot\varphi\csc\varphi\left(-\csc^2(\varphi)\varphi_1\varphi_2 + \cot(\varphi)\varphi_{12} + \csc^2(\varphi)\varphi_1\varphi_2\right)\\
&\qquad + \csc^2\varphi\left(\cot(\varphi)\csc(\varphi)\varphi_1\varphi_2 - \csc(\varphi)\varphi_{12} - \cot(\varphi)\csc(\varphi)\varphi_1\varphi_2\right)\\
&= \left(\cot^2\varphi\csc\varphi - \csc^3\varphi\right)\varphi_{12}\\
&= \csc(\varphi)\varphi_{12}.
\end{aligned}$$

6. The curvature in orthogonal coordinates.

A local coordinate system is called *orthogonal* if the cross term $g_{12} = 0$ on all of Ω. One such system, the lines of curvature coordinates, shall be described in Section 8. There are several other coordinate systems with this property, for example the geodesic polar coordinates, the Fermi coordinates or isothermal coordinates. The first two require knowledge about geodesic curves on the surface; the third is deeper requiring solution of the Beltrami equations on the surface.

Let us assume that $g_{12} = 0$ on all of our coordinate patch Ω. Thus also $g^{12} = 0$. Then using (12), the Christoffel symbols take the form

$$\begin{aligned}
\Gamma_{11}{}^1 &= \tfrac12 g^{11} g_{11,1} + g^{12}\left(g_{12,1} - \tfrac12 g_{11,2}\right) &&= \frac{g_{11,1}}{2g_{11}}\\
\Gamma_{21}{}^1 = \Gamma_{12}{}^1 &= \tfrac12 g^{11}\, g_{11,2} + \tfrac12 g^{12}\, g_{22,1} &&= \frac{g_{11,2}}{2g_{11}}\\
\Gamma_{22}{}^1 &= g^{11}\left(g_{21,2} - \tfrac12 g_{22,1}\right) + \tfrac12 g^{12}\, g_{22,2} &&= -\frac{g_{22,1}}{2g_{11}}\\
\Gamma_{11}{}^2 &= \tfrac12 g^{21} g_{11,1} + g^{22}\left(g_{12,1} - \tfrac12 g_{11,2}\right) &&= -\frac{g_{11,2}}{2g_{22}}\\
\Gamma_{21}{}^2 = \Gamma_{12}{}^2 &= \tfrac12 g^{21}\, g_{11,2} + \tfrac12 g^{22}\, g_{22,1} &&= \frac{g_{22,1}}{2g_{22}}\\
\Gamma_{22}{}^2 &= g^{21}\left(g_{21,2} - \tfrac12 g_{22,1}\right) + \tfrac12 g^{22}\, g_{22,2} &&= \frac{g_{22,2}}{2g_{22}}.
\end{aligned}$$

Thus, the curvature (11) becomes

$$\begin{aligned}
K =& \frac{1}{g_{11}}\left(\frac{\partial}{\partial u^2}{\Gamma_{11}}^2 - \frac{\partial}{\partial u^1}{\Gamma_{12}}^2 + {\Gamma_{11}}^1\,{\Gamma_{12}}^2 + {\Gamma_{11}}^2\,{\Gamma_{22}}^2 - {\Gamma_{12}}^1\,{\Gamma_{11}}^2 - {\Gamma_{12}}^2\,{\Gamma_{21}}^2\right)\\
=& \frac{1}{2g_{11}}\left(-\frac{\partial}{\partial u^2}\left(\frac{g_{11,2}}{g_{22}}\right) - \frac{\partial}{\partial u^1}\left(\frac{g_{22,1}}{g_{22}}\right) + \frac{g_{11,1}\,g_{22,1}}{2g_{11}\,g_{22}} - \frac{g_{11,2}\,g_{22,2}}{2g_{22}^2} + \frac{{g_{11,2}}^2}{2g_{11}\,g_{22}} - \frac{{g_{22,1}}^2}{2{g_{22}}^2}\right)\\
=& -\frac{1}{2g_{11}}\left(\frac{\partial}{\partial u^1}\left(\frac{g_{22,1}}{g_{22}}\right) - \frac{g_{11,1}\,g_{22,1}}{2g_{11}\,g_{22}} + \frac{{g_{22,1}}^2}{2{g_{22}}^2}\right) - \frac{1}{2g_{11}}\left(\frac{\partial}{\partial u^2}\left(\frac{g_{11,2}}{g_{22}}\right) + \frac{g_{11,2}\,g_{22,2}}{2g_{22}^2} - \frac{{g_{11,2}}^2}{2g_{11}\,g_{22}}\right)
\end{aligned}$$

Observing that

$$\begin{aligned}
\frac{\partial}{\partial u^1}\left(\frac{g_{22,1}}{g_{22}}\right) =& \frac{\partial}{\partial u^1}\left(\frac{g_{22,1}}{\sqrt{g_{11}g_{22}}}\sqrt{\frac{g_{11}}{g_{22}}}\right) = \frac{\partial}{\partial u^1}\left(\frac{g_{22,1}}{\sqrt{g_{11}g_{22}}}\right)\sqrt{\frac{g_{11}}{g_{22}}} + \frac{g_{22,1}}{\sqrt{g_{11}g_{22}}}\frac{\partial}{\partial u^1}\left(\sqrt{\frac{g_{11}}{g_{22}}}\right)\\
=& \frac{\partial}{\partial u^1}\left(\frac{g_{22,1}}{\sqrt{g_{11}g_{22}}}\right)\sqrt{\frac{g_{11}}{g_{22}}} + \frac{g_{11,1}\,g_{22,1}}{2g_{11}\,g_{22}} - \frac{{g_{22,1}}^2}{2{g_{22}}^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial u^2}\left(\frac{g_{11,2}}{g_{22}}\right) =& \frac{\partial}{\partial u^2}\left(\frac{g_{11,2}}{\sqrt{g_{11}\,g_{22}}}\sqrt{\frac{g_{11}}{g_{22}}}\right) = \frac{\partial}{\partial u^2}\left(\frac{g_{11,2}}{\sqrt{g_{11}\,g_{22}}}\right)\sqrt{\frac{g_{11}}{g_{22}}} + \frac{g_{11,2}}{\sqrt{g_{11}\,g_{22}}}\frac{\partial}{\partial u^2}\left(\sqrt{\frac{g_{11}}{g_{22}}}\right)\\
=& \frac{\partial}{\partial u^2}\left(\frac{g_{22,1}}{\sqrt{g_{11}\,g_{22}}}\right)\sqrt{\frac{g_{11}}{g_{22}}} + \frac{{g_{11,2}}^2}{2g_{11}\,g_{22}} - \frac{g_{11,2}\,g_{22,2}}{2{g_{22}}^2}
\end{aligned}$$

we obtain

$$K = -\frac{1}{2\sqrt{g_{11}\,g_{22}}}\left[\frac{\partial}{\partial u^1}\left(\frac{g_{22,1}}{\sqrt{g_{11}g_{22}}}\right) + \frac{\partial}{\partial u^2}\left(\frac{g_{11,2}}{\sqrt{g_{11}\,g_{22}}}\right)\right]. \tag{12}$$

7. Example: the curvature of a sphere.

To illustrate formula (12), consider the usual latitude-longitude coordinates of the unit sphere

$$X(u^1, u^2) = \left(\cos u^1 \cos u^2, \cos u^1\, \sin u^2, \sin u^1\right).$$

Then the first fundamental form is found by

$$\begin{gathered}
X_1 = \left(-\sin u^1 \cos u^2, -\sin u^1\, \sin u^2, \cos u^1\right)\\
X_2 = \left(-\cos u^1 \sin u^2, \cos u^1\, \cos u^2, 0\right)\\
g_{11} = X_1 \cdot X_1 = 1, \qquad g_{12} = X_1 \cdot X_2 = 0, \qquad g_{22} = \cos^2 u^1,
\end{gathered}$$

making (u^1, u^2) an orthogonal coordinate system. Hence

$$\sqrt{g_{11}\,g_{22}} = \cos u^1, \qquad g_{22,1} = -2\cos u^1\, \sin u^1, \qquad g_{11,2} = 0.$$

Thus the formula (12) gives

$$K = -\frac{1}{2\cos u^1}\frac{\partial}{\partial u^1}\left(\frac{-2\cos u^1\, \sin u^1}{\cos u^1}\right) = 1.$$

8. Lines of curvature coordinates.

We sketch the argument that in the neighborhood of a non-umbillic point $P \in M$, it is possible to find a change of coordinates such that in these new coordinates, both $h_{12} = 0$ and $g_{12} = 0$ near P.

Lemma. [Lines of curvature coordinates.] *Suppose that a point $P \in M$ of a smooth regular surface in three space is not an umbillic (the principal curvatures satisfy $k_1(P) \neq k_2(P)$.) Then there is a coordinate patch $X : \Omega' \to M$ in the neighborhood of $P \in X(\Omega')$ which has the following properties. Denoting the patch by $X(z^1, z^2)$ for $(z^1, z^2) \in \Omega'$, the coordinate lines are lines of curvature, which means that they go in the principal directions. This means that $X_i(z^1, z^2) = f_i(z^1, z^2)V_i(z^1, z^2)$, where $f_i > 0$ are positive functions and V_i are the principal directions corresponding to k_i, namely the eigenvector V_i of the shape operator $S(V_i) = k_i V_i$. Moreover, the first and second fundamental forms satisfy $g_{12} = h_{12} = 0$ on all of Ω'.*

Sketch of the argument. Choose any patch in the neighborhood of $P : X(u^1, u^2) : \Omega \to M$ where $P \in X(\Omega)$. Since the principal curvatures are continuous functions, there is a possibly smaller subneighborhood $\Omega'' \subset \Omega$ with $P \in X(\Omega'')$ consisting entirely of non-umbillic points, namely such that $k_1 < k_2$ for all points of Ω''. The corresponding unit tangent vector fields V_i may be chosen to be well-defined continuous functions in Ω''. Define new vector functions $W_j(t; u_0^1, u_0^2) \in \Omega''$ by solving the initial value problem for the system of ordinary differential equations

$$\begin{aligned} \frac{\partial}{\partial t} X(W_j(t)) =& V_j(X(W_j(t))), \\ W_j(0) =& (u_0^1, u_0^2). \end{aligned}$$

The curves $t \mapsto X(W_j(t; u_0^1, u_0^2))$ are lines of curvature since they are in the V_j direction. When $t = 0$ then $W_j(0) = (u_0^2, u_0^2) \in \Omega''$ is the initial point. There are two families of curves that foliate Ω'', corresponding to the two principal directions. The idea of the argument is to make these families of curves the new coordinate curves.

Suppose $X(u_0^1, u_0^2) = P$ is the center point of the patch. Through this point are curves from each of the families. Call them $\sigma(z^1) = W_1(z^1; u_0^1, u_0^2)$ and $\tau(z^2) = W_2(z^2; u_0^1, u_0^2)$. Thus $\sigma(0) = \tau(0) = (u_0^1, u_0^2)$. Now the idea is the following. Any point $Q = (u^1, u^2)$ sufficiently close to the center lies on exactly one integral curve from each family. This curve from the first family must cross τ at exactly one point, say $\tau(z^2)$. Similarly, the curve through Q from the second family must cross σ at exactly one point, namely $\sigma(z^1)$. The desired change of variables is then given by the mapping just described $F : (u^1, u^2) \mapsto (z^1, z^2)$. This is a smooth mapping because the solutions of differential equations depend smoothly on initial points. The mapping is invertible at the origin. One can check that $dF = I$ at (u_0^1, u_0^2). By the implicit function theorem, there is a neighborhood $(u_0^1, u_0^2) \in \Omega' \subset \Omega''$ on which F admits a smooth inverse function $(u^1, u^2) = F^{-1}(z^1, z^2)$. The desired chart is given by $X(z^1, z^2) = X(F^{-1}(z^1, z^2))$.

The rest of the argument is to verify the claims. The coordinate lines are in the principal directions, since they follow the trajectories of the ODE. $g_{12} = X_1 \cdot X_2 = 0$ because the principal directions are orthogonal. In these coordinates, $k_1 X_1 = S(X_1) = -h_{1j} g^{jk} X_k$ so that the X_2 term is $0 = -h_{1j} g^{j2} = -h_{12} g^{22}$ because $g^{12} = 0$ which implies $h_{12} = 0$.

References

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