Math 3220 § 2. **Third Midterm Exam** Name: Solutions
Treibergs April 1, 2020

1. *Suppose $a \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to R$ is a function all of whose second partial derivatives exist and are continuous. Prove that there are positive numbers M and r such that*

$$|f(x) - f(a) - df(a)(x-a)| \le M\|x-a\|^2 \qquad \text{for all } x \in \mathbb{R}^n \text{ such that } |x-a| < r.$$

This is an application of Taylor's Formula.

Choose $r > 1$, say $r = 1$. First, let's get a bound on the second derivatives in the closed r-ball $\overline{B}_r(a) \subset \mathbb{R}^n$ about a,

$$M = \sup_{x \in \overline{B}_r(a)} \|d^2 f(x)\|_2,$$

where $\|L_{ij}\|_2^2 = \sum_{i,j} \ell_{i,j}^2$ denotes L^2 norm of the matrix L. The L^2 norm is polynomial in L so $\|d^2 f(x)\|_2$ is a continuous function which is bounded on the compact set $\overline{B}_r(a)$.

Second, apply Taylor's formula up to the first order with remainder. This requires that f be differentiable up to second order in $B_r(a)$, which follows from the assumption that all second order partial derivatives exist and are continuous in $B_r(a)$. For $x \in B_r(a)$ we have

$$f(x) - f(a) - df(a)(x-a) = \frac{1}{2} d^2 f(c)(x-a)^2$$

where c is a point on the line segment from a to x so $c \in \overline{B}_r(a)$. Taking absolute values, we have

$$|f(x) - f(a) - df(a)(x-a)| = \left| \frac{1}{2} d^2 f(c)(x-a)^2 \right| \le \frac{1}{2} M \|x-a\|^2,$$

as to be shown. The last inequality follows from a matrix inequality. If L is an $n \times n$ matrix and $v \in \mathbb{R}^n$ then by the Schwartz Inequality and operator norm inequality

$$|L\, v^2| = |v \cdot Lv| \le \|v\|\, \|Lv\| \le \|v\|\, \|L\|\, \|v\| \le \|L\|_2\, \|v\|^2,$$

where $\|L\|$ is the operator norm.

2. *Let $E = \{(t,t) \in \mathbb{R}^2 : t \in \mathbb{Q} \text{ and } 0 \le t \le 1\}$ be the set consisting rational points of the line segment from $(0,0)$ to $(1,1)$. What is The upper volume $\overline{V}(E)$ of E? For each $\epsilon > 0$, describe a partition $\mathcal{P}$ such that*

$$U(\chi_E, \mathcal{P}) < \overline{V}(E) + \epsilon.$$

Is E a Jordan Region? Explain.

$\overline{V}(E) = 0$. Consider the partition P_n of $R = [0,1] \times [0,1]$ into n^2 subsquare of sides $\frac{1}{n}$. If the partition coordinates are $x_i = y_i = \frac{i}{n}$ for $i = 0, \ldots, n$ then denote the subsquare $p_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i, j = 1, \ldots, n$. Since the set E is located on the diagonal, only the subsquares that touch the diagonal contain points of E. These are the diagonal squares p_{ij} where $i = j$ and the off-diagonal squares p_{ij} where $|i-j| = 1$. Note that the diagonal corners of the off-diagonal squares are rational points, so they are all members of E. The characteristic function takes values $\chi_E \subset \{0, 1\}$. Thus we find

$$m_{ij} = \inf_{p_{ij}} \chi_E = 0; \qquad M_{ij} = \sup_{p_{ij}} \chi_E = \begin{cases} 1, & \text{if } |i-j| \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

because all subsquares contain points not in E, and only the diagonal and off diagonal squares contain points of E.

Thus the sums may be computed

$$\begin{aligned} L(\chi_E, P_n) &= \sum_{ij} m_{ij}\, dV(p_{ij}) = 0, \\ U(\chi_E, P_n) &= \sum_{ij} M_{ij}\, dV(p_{ij}) \\ &= \sum_{i=1}^{n} \sum_{j, |j-i| \le 1} 1 \cdot \frac{1}{n^2} \\ &\le \frac{3}{n^2} \sum_{i=1}^{n} 1 = \frac{3n}{n^2} = \frac{3}{n}. \end{aligned}$$

because there are at most three j's that satisfy $|j - i| \le 1$ for every i. It follows that for every $\epsilon > 0$, for $n > \frac{3}{\epsilon}$ we have a partition P_n such that

$$U(\chi_E, P_n) - L(\chi_E, P_n) \le \frac{3}{n} < \epsilon.$$

By the principal integrability condition, χ_E is integrable so E is a Jordan Region. We also have

$$0 \le L(\chi_E, P_n) \le V(E) \le \overline{V}(E) = \overline{\int_R} \chi_E(x)\, dV(x) \le U(\chi_E, P_n) \le \frac{3}{n}$$

so that as $n \to \infty$, we see that $\overline{V}(E) = 0$. Another reason E is integrable is that it's upper volume is zero. It is a set of volume zero, so a Jordan Region.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement. *Let $\mathcal{D} \subset \mathbf{R}^2$ be an open set and $f(x, y) : \mathcal{D} \to \mathbb{R}^2$ be a map all of whose first partial derivatives exist and are continuous on $\mathcal{D}$. If $df(x, y)$ is in invertible for all $(x, y) \in \mathcal{D}$ then there exists an $\mathcal{C}^1$ inverse function $f^{-1} : f(\mathcal{D}) \to \mathcal{D}$.*

   False. The condition only guarantees local invertibility by the Inverse Function Theorem. If we write the function

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix},$$

   which is $f(z) = z^2$ where $z = x + iy$, we get a function whose Jacobian matrix is everywhere nonsingular on the punctured plane $\mathcal{D} = \mathbb{R}^2 \backslash \{(0, 0)\}$ but is not one to one, so not globally invertible. Indeed, if $(x, y) \ne (0, 0)$,

$$\det(dF(x, y)) = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) > 0$$

   and

$$F\begin{pmatrix} 0 \\ 1 \end{pmatrix} = F\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

(b) STATEMENT. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (e^x + y + y^3, e^x - y - y^3)$. Then $f(W)$ is open for every open $W \subset \mathbb{R}^2$.*

TRUE. Computing the Jacobian determinant we find

$$\begin{vmatrix} e^x & 1+3y^2 \\ e^x & -1-3y^2 \end{vmatrix} = -2e^{2x}(1+3y^2) < 0$$

for all $(x,y) \in \mathbb{R}^2$. The map f is smooth defined on the open set $V = \mathbb{R}^2$. Thus f is an open map follows from the Open Mapping Theorem.

Open Mapping Theorem. Let $U \subset \mathbb{R}^p$ be an open set and $f : U \to \mathbb{R}^p$ be a $\mathcal{C}^1$ map such that f is nonsingular at every point of an open subset $V \subset U$, then $f : V \to \mathbb{R}^p$ is an open map.

(c) STATEMENT. *Let f and g be bounded functions on the aligned rectangle $R \subset \mathbb{R}^d$. Then the upper integrals satisfy $\overline{\int_R}(f+g) = \overline{\int_R} f + \overline{\int_R} g$.*

FALSE. Let $R = [0,1] \times [0,1]$,

$$f(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \mathbb{Q} \times \mathbb{Q} \text{ are rational points;} \\ 0, & \text{otherwise.} \end{cases}$$

and $g(x,y) = 1 - f(x,y)$. Then $f(x,y) + g(x,y) = 1$ for all (x,y). This gives

$$1 = \overline{\int_R}(f+g) \neq \overline{\int_R} f + \overline{\int_R} g = 1 + 1$$

because $M_{ij} = \sup_{p_{ij}} h = 1$ for every subrectangle with $V(p_{ij}) > 0$ for every partition P and for $h = f$, $h = g$ or $h = f + g$.

4. (a) *Complete the statement of the following theorem:*

Theorem. *Let $R \subset \mathbb{R}^2$ be an aligned rectangle and $f : R \to \mathbb{R}$ be a bounded function. Then f is integrable on R if and only if .....*

Using just your theorem, prove that $f(x_1, x_2) = 1 + x_1 + 2x_2$ is integrable on $R =$ $[0,1] \times [0,1]$.

> **Theorem.** Let $R \subset \mathbb{R}^2$ be an aligned rectangle and $f : R \to \mathbb{R}$ be a bounded function. Then f is *integrable* on R if and only if there exists a sequence of partitions $\mathcal{P}_n$ such that
>
> $$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) \to 0 \qquad \text{as } n \to \infty.$$

Consider the partition P_n of R into n^2 subsquare of sides $\frac{1}{n}$. If the partition coordinates are $x_{i,1} = x_{i,2} = \frac{i}{n}$ for $i = 0, \ldots, n$ then denote the subsquare $p_{ij} = [x_{i-1,1}, x_{i,1}] \times [x_{j-1,2}, x_{j,2}]$ for $i, j = 1, \ldots, n$. Since $\nabla f(x_1, x_2) = (1, 2)$ for all $(x_1, x_2) \in R$ the function is increasing in the northeast direction so the maximum of f occurs in the northeast corner of p_{ij} and the minimum in the southwest corner. Thus for any $i, j = 1, \ldots, n$ we have

$$m_{ij} = \inf_{p_{ij}} f = 1 + x_{i-1,1} + 2x_{j-1,2}; \qquad M_{ij} = \sup_{p_{ij}} f = 1 + x_{i,1} + 2x_{j,2}$$

It follows that

$$\begin{aligned} M_{ij} - m_{ij} &= (1 + x_{i,1} + 2x_{j,2}) - (1 + x_{i-1,1} + 2x_{j-1,2}) \\ &= (x_{i,1} - x_{i-1,1}) + 2\,(x_{j,2} - x_{j-1,2}) \end{aligned}$$

Computing the sum,

$$
\begin{aligned}
U(f, P_n) - L(f, P_n) &= \sum_{ij} \left[M_{ij} - m_{ij}\right] dV(p_{ij}) \\
&= \sum_{ij} \left[(x_{i,1} - x_{i-1,1}) + 2\,(x_{j,2} - x_{j-1,2}) \right] \frac{1}{n^2} \\
&= \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} (x_{i,1} - x_{i-1,1}) + \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_{j,2} - x_{j-1,2}) \\
&= \frac{1}{n^2} \sum_{j=1}^{n} (x_{n,1} - x_{0,1}) + \frac{2}{n^2} \sum_{i=1}^{n} (x_{n,2} - x_{0,2}) \\
&= \frac{1}{n^2} \sum_{j=1}^{n} (1 - 0) + \frac{2}{n^2} \sum_{i=1}^{n} (1 - 0) \\
&= \frac{n}{n^2} + \frac{2n}{n^2} = \frac{3}{n},
\end{aligned}
$$

since the inside sums telescoped. It follows that

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) \to 0 \qquad \text{as } n \to \infty.$$

Hence, by the boxed theorem, f is integrable.

5. *Let* $$F(x, y, z, u, v) = (x - 2z + 3u - 4v,\ x + yz + uv).$$
Prove that the level set $\mathcal{S} = \{(x, y, z, u, v) : F(x, y, z, u, v) = (-3, 9)\}$ *is a locally parameterized surface near the point* $P = (1, 2, 3, 2, 1) \in \mathcal{S}$. *[Hint: show that near the point* $\mathcal{S}$ *is the graph of some functions and nothing more.] What is the tangent space to* $\mathcal{S}$ *at* $(1, 2, 3, 2, 1)$*?*
The map $f : \mathcal{D} = \mathbb{R}^5 \to \mathbb{R}^2$ is $\mathcal{C}^1$ since it is polynomial. Its Jacobian is

$$df(x, y, z, u, v) = \begin{pmatrix} 1 & 0 & -2 & 3 & -4 \\ 1 & z & y & v & u \end{pmatrix}, \qquad df(1, 2, 3, 2, 1) = \begin{pmatrix} 1 & 0 & -2 & 3 & -4 \\ 1 & 3 & 2 & 1 & 2 \end{pmatrix}.$$

The determinant of the last two columns

$$\det\left(\frac{\partial(f_1, f_2)}{\partial(u, v)}\right) = \begin{vmatrix} 3 & -4 \\ 1 & 2 \end{vmatrix} = 10$$

is nonzero. Thus we may apply the Implicit Function Theorem which says that if for open $\mathcal{D} \subset \mathbb{R}^5$ there is a $\mathcal{C}^1$ map $f(x, y, z, u, v) : \mathcal{D} \to \mathbb{R}^2$ such that at the point $P = (x_0, y_0, z_0, u_0, v_0) \in \mathcal{D}$ the Jacobian $\dfrac{\partial(f_1, f_2)}{\partial(u, v)}(P)$ is nonsingular, then there is an open set $W \in \mathcal{D}$ such that $P \in W$ and an open set $V \in \mathbb{R}^3$ such that $(x_0, y_0, z_0) \in V$ and a $\mathcal{C}^1$ function $g = (g_1, g_2) : V \to \mathbb{R}^2$ such that $G(x, y, z) = (x, y, z, g_1(x, y, z), g_2(x, y, z)) \in W$ for all $(x, y, z) \in V$, $g(x_0, y_0, z_0) = (u_0, v_0)$ and

$$(x, y, z, u, v) \in \mathcal{S} \cap W \qquad \Longleftrightarrow \qquad (u, v) = g(x, y, z) \text{ for some } (x, y, z) \in V.$$

Hence, $\mathcal{S}$ is locally (in W) a $\mathcal{C}^1$ parameterized three dimensional surface. In fact, $G : V \to W$ is the parameterization.

We know that the tangent space at P is $dG(x_0, y_0, z_0)(\mathbb{R}^3) = \ker df(x_0, y_0, z_0, u_0, v_0)$ translated to P. Computing the null space, we find by subtracting the first row from the second that

$$\begin{pmatrix} 1 & 0 & -2 & 3 & -4 \\ 1 & 3 & 2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 3 & -4 \\ 0 & 3 & 4 & -2 & 6 \end{pmatrix}.$$

The last three columns are free. Setting $z = r$, $u = s$ and $v = t$ where r, s and t are arbitrary, we solve for x and y in terms of r, s and t in the homogeneous equation. Translating to P we get that the tangent space is

$$T_P(\mathcal{S}) = P + \ker df(P) = \left\{ \begin{pmatrix} 1 + 2r - 3s + 4t \\ 2 + \frac{1}{3}(-4r + 2s - 6t) \\ 3 + r \\ 2 + s \\ 1 + t \end{pmatrix} : r, s, t \in \mathbb{R} \right\}.$$