Math 3220 § 2. **Second Midterm Exam** Name: Solutions
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1. *Let* $f(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{x^2+y^2}, & \text{if } (x,y) \ne (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

Is f is continuous at $(0,0)$? Why?

Do the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$? Why?

Is f differentiable at $(0,0)$? Why?

f is continuous at $(0,0)$. Choose $\epsilon > 0$. Let $\delta = \epsilon$. For any $(x,y) \in \mathbf{R}^2$ such that $0 < \|(x,y) - (0,0)\| < \delta$ we have

$$\begin{aligned} |f(x,y) - f(0,0)| = \left| \frac{x^3 - xy^2}{x^2+y^2} - 0 \right| &= \frac{|x^2 - y^2|\,|x|}{x^2+y^2} \le \frac{(|x^2| + |y^2|)|x|}{x^2+y^2} \\ &= |x| \le \sqrt{x^2+y^2} = \|(x,y) - (0,0)\| < \delta = \epsilon. \end{aligned}$$

Thus $f(x,y) \to 0 = f(0,0)$ as $(x,y) \to (0,0)$ so f is continuous at $(0,0)$.

The partial derivatives exist everywhere. At the origin, since $f(x,0) = x$ we have $f_x(0,0) = 1$ and since $f(0,y) = 0$ we have $f_y(0,0) = 0$. If $(x,y) \ne (0,0)$ then $f(x,y)$ is a quotient of polynomials whose denominator is not zero, hence admits partial derivatives everywhere.

f is not differentiable at $(0,0)$. If it were, then the differential would be $df(0,0)(h,k) = f_x(0,0)h + f_y(0,0)k = h$. But the corresponding difference quotient does not converge to zero. Indeed, if $(h,k) \ne (0,0)$,

$$\begin{aligned} q(h,k) = \frac{f(0+h,0+k) - f(0,0) - df(0,0)(h,k)}{\|(0+h,0+k)\|} &= \frac{\dfrac{h^3 - hk^2}{h^2+k^2} - h}{\sqrt{h^2+k^2}} \\ &= \frac{h^3 - hk^2 - \left(h^2+k^2\right)h}{\left(h^2+k^2\right)^{3/2}} \\ &= \frac{-2hk^2}{\left(h^2+k^2\right)^{3/2}} \end{aligned}$$

$q(t,0) = 0$ and $q(t,t) = -\dfrac{1}{\sqrt{2}}$ so $\lim\limits_{t\to 0} q(t,0) = 0$ does not equal $\lim\limits_{t\to 0} q(t,t) = -\dfrac{1}{\sqrt{2}}$. Thus $q(h,k)$ does not have a limit as $(h,k) \to (0,0)$ so f is not dfferentiable at $(0,0)$.

2. *Let* $f : \mathbf{R}^2 \to \mathbf{R}^2$ *be a function. Define:* f *is* continuous *on* $\mathbf{R}^2$. *Define:* $E \subset \mathbf{R}^2$ *is a* closed set. *Using only your definitions, prove that the pull-back* $f^{-1}(E)$ *is a closed set whenever* $E \subset \mathbf{R}^2$ *is a closed set.*

   f is *continuous on* $\mathbf{R}^2$ if for every $\mathbf{a} \in \mathbf{R}^2$ and for every $\epsilon > 0$ there is a $\delta > 0$ so that if $\mathbf{x} \in \mathbf{R}^2$ is any point such that $\|\mathbf{x} - \mathbf{a}\| < \delta$ then $\|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon$.

   $E \subset \mathbf{R}^2$ is *closed* if the complement $E^c = \mathbf{R}^2 \backslash E$ is open. E^c is open if for every $\mathbf{b} \in E^c$ there is an $\epsilon > 0$ such that the open ball about $\mathbf{b}$ of radius ϵ is in E^c, that is $B_\epsilon(\mathbf{b}) \subset E^c$.

   Let $E \subset \mathbf{R}^2$ be a closed set. To show that $f^{-1}(E)$ is closed we show its complement $(f^{-1}(E))^c$ is open, according to our definition. Choose a point $\mathbf{a} \in (f^{-1}(E))^c$. Then $\mathbf{b} = f(\mathbf{a}) \in E^c$. Since E^c is open, there is $\epsilon > 0$ such that $B_\epsilon(\mathbf{b}) \subset E^c$. By continuity, there is $\delta > 0$ so that $\|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon$ whenever $\mathbf{x} \in \mathbf{R}^2$ is any point such that $\|\mathbf{x} - \mathbf{a}\| < \delta$. In other words $f(B_\delta(\mathbf{a})) \subset B_\epsilon(\mathbf{b}) \subset E^c$. It follows that $B_\delta(\mathbf{a}) \subset (f^{-1}(E))^c$. Thus every point of the complement $\mathbf{a} \in (f^{-1}(E))^c$ is surrounded by a $\delta > 0$ ball entirely in the complement $B_\delta(\mathbf{a}) \subset (f^{-1}(E))^c$ which is the definition that $(f^{-1}(E))^c$ is open, hence its complement $f^{-1}(E)$ is closed.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT. *Let* $f(x, y) : \mathbf{R}^2 \to \mathbf{R}$ *be a function such that the partial derivatives* $\frac{\partial f}{\partial x}(x, y)$ *and* $\frac{\partial f}{\partial y}(x, y)$ *exist and are continuous for all* (x, y). *If* $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ *exists then* $\frac{\partial^2 f}{\partial y \partial x}(0, 0)$ *exists and satisfies* $\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial^2 f}{\partial y \partial x}(0, 0)$.

   FALSE. If we knew in addition that $f_{yx}(x, y)$ existed in a neighborhood of the origin and was continuous at $(0, 0)$ then f_{xy} would exist at $(0, 0)$ and $f_{xy}(0, 0) = f_{xy}(0, 0)$. To answer the question, we must provide a counterexample. The function

   $$f(x, y) = \begin{cases} \dfrac{x^3 y - x y^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

   has partial derivatives at all points, but the second cross partials are not equal at the origin. We have $f(x, 0) = 0$ and $f(0, y) = 0$ so $f_x(0, 0) = f_y(0, 0) = 0$. For $(x, y) \neq (0, 0)$,

   $$f_x(x, y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}, \qquad f_y(x, y) = \frac{x^5 - 4x^3 y^2 - x y^4}{(x^2 + y^2)^2}.$$

   These are rational functions thus are continuous away from the origin where the denominators vanish. Since the numerators are bounded by $6(x^2 + y^2)^{5/2}$ we see that $f_x(x, y)$ and $f_y(x, y)$ tend to zero as $(h, k) \to (0, 0)$ thus are continuous at the origin also. Also, $f_x(0, y) = -y$ and $f_y(x, 0) = x$ so $f_{xy}(0, 0) = -1 \neq 1 = f_{yx}(0, 0)$. Thus we have the desired counterexample: for f, both f_x and f_y are continuous everywhere and $f_{yx}(0, 0)$ exists, but, although $f_{xy}(0, 0)$ exists, it does not equal $f_{yx}(0, 0)$.

   (b) STATEMENT. *Let* $\overline{B_1(0, 0)}$ *be the closed ball in the plane and* $f : \overline{B_1(0, 0)} \to \mathbf{R}$ *be continuous. Then* $f\left(\overline{B_1(0, 0)}\right)$ *is a closed and bounded interval.*

   TRUE. $K = \overline{B_1(0, 0)}$ is a closed and bounded set, therefore compact. A continuous function maps a compact set to a compact set. Also, K is convex, therefore connected. A continuous function also maps a connected set to a connected set. Therefore, $f(K)$ is a compact connected subset of $\mathbf{R}$, hence it is a closed, bounded interval.

(c) STATEMENT. *Let $f_n : \mathbf{R}^2 \to \mathbf{R}$ be functions such that $f_n \to 0$ pointwise on $\mathbf{R}^2$. Then $f_n(\mathbf{x}_n) \to 0$ as $n \to \infty$ for every sequence $\{\mathbf{x}_n\}$ of $\mathbf{R}^2$.*

FALSE. The functions $\dfrac{1}{1+x^2+(y-n)^2}$ converge to zero pointwise but not unifolrmly. Fix $(x,y) \in \mathbf{R}^2$.

$$\lim_{n\to\infty} f_n(x,y) = \lim_{n\to\infty} \frac{1}{1+x^2+(y-n)^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{x^2}{n^2}+\left(\frac{y}{n}-1\right)^2}$$
$$= \frac{0}{0+0+(0+1)^2} = 0.$$

But for the sequence $\mathbf{x}_n = (0,n)$ we have $f_n(\mathbf{x}_n) = f_n(0,n) = 1$ so $f_n(\mathbf{x}_n)$ does not converge to zero as $n \to \infty$.

4. *Let $\mathbf{a}, \mathbf{h} \in \mathbf{R}^3$ For the given $f : \mathbf{R}^2 \to \mathbf{R}^3$ and $g : \mathbf{R}^3 \to \mathbf{R}^2$, compute $d(f \circ g)(\mathbf{a})(\mathbf{h})$ in two ways, directly and using the chin rule.*

$$\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} h \\ k \\ \ell \end{pmatrix}, \quad f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u+3v \\ v^2 \\ uv \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ yz \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Let

$$\mathbf{b} = g(\mathbf{a}) = g\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0+1 \\ 1\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The differentials are given by the Jacobian matrices

$$df\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 2v \\ v & u \end{pmatrix} \quad \text{so} \quad df(\mathbf{b}) = df\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \\ 2 & 1 \end{pmatrix}$$

and

$$dg\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & z & y \end{pmatrix} \quad \text{so} \quad dg(\mathbf{a}) = dg\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

It follows that

$$d(f\circ g)(\mathbf{a}) = df(\mathbf{b})dg(\mathbf{a}) = \begin{pmatrix} 1 & 3 \\ 0 & 4 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 3 \\ 0 & 8 & 4 \\ 2 & 4 & 1 \end{pmatrix}$$

The composite function is

$$f \circ g \begin{pmatrix} x \\ y \\ z \end{pmatrix} = f\left(g \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = f \begin{pmatrix} x+y \\ yz \end{pmatrix} = \begin{pmatrix} (x+y)+3yz \\ (yz)^2 \\ (x+y)yz \end{pmatrix} = \begin{pmatrix} x+y+3yz \\ y^2z^2 \\ xyz+y^2z \end{pmatrix}.$$

Its differential is

$$d(f\circ g) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1+3z & 3y \\ 0 & 2yz^2 & 2y^2z \\ yz & xz+2yz & xy+y^2 \end{pmatrix}$$

so

$$d(f\circ g)(\mathbf{a}) = d(f\circ g) \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 3 \\ 0 & 8 & 4 \\ 2 & 4 & 1 \end{pmatrix}.$$

Both are the same, thus

$$d(f\circ g)(\mathbf{a})(\mathbf{h}) = df(\mathbf{b})dg(\mathbf{a})(\mathbf{h}) = \begin{pmatrix} 1 & 7 & 3 \\ 0 & 8 & 4 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} h \\ k \\ \ell \end{pmatrix}.$$

5. *Let* $f(x,y) = \sum_{k=0}^{\infty} \frac{\sin(nxy)}{1+n^3}$. *Answer the following questions and provide a proof.*

   Does $f(x,y)$ *converge pointwise on* $\mathbf{R}^2$*?*

   Does $f(x,y)$ *converge uniformly on* $\mathbf{R}^2$*?*

   Is $f(x,y)$ *continuous on* $\mathbf{R}^2$*?*

   We will show that $f(x,y)$ converges uniformly, thus as a consequence, it converges pointwise on $\mathbf{R}^2$.

   The uniform convergence follows from the Weierstrass M-test. Indeed, we see that the summands satisfy for all $(x,y) \in \mathbf{R}^2$,

$$\left| \frac{\sin(nxy)}{1+n^3} \right| \leq \frac{1}{1+n^3} = b_n,$$

   where

$$\sum_{n=1}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{1+n^3} < \infty$$

by the integral test: $\displaystyle\int_0^\infty \frac{dx}{1+x^3} < \infty$. So the partial sums

$$S_m(x,y) = \sum_{n=0}^{m} \frac{\sin(nxy)}{1+n^3}$$

converge $S_m(x,y) \to f(x,y)$ uniformly on $\mathbf{R}^2$ as $m \to \infty$.

The continuity of $f(x,y)$ follows from the uniform convergence. The partial sums $S_m(x,y)$ are all continuous since they are finite trigonometric sums of smooth weighted sine functions $\sin(nxy)$. The uniform limit of continuous functions is continuous, hence $f(x,y)$ is continuous on $\mathbf{R}^2$.