Math 3220 § 2. **First Midterm Exam** Name: Solutions
Treibergs January 29, 2020

1. *Let $F, G \subset \mathbf{R}^d$. Define: G is an* open set. *Define: F is a* closed set. *Using just your definitions, show that the line $L = \{(x,y) \in \mathbf{R}^2 : x = y\}$ is closed.*

   G is open if for every $\mathbf{x} \in G$ there is $r > 0$ so that $B_r(\mathbf{x}) \subset G$, where the open ball of radius r about $\mathbf{x}$ is given by $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbf{R}^d : \|\mathbf{x} - \mathbf{y}\| \le r\}$.

   F is closed if its complement F^c is open.

   To show L is closed, we show that its complement L^c is open. Choose $(x,y) \in L^c$. Then $x \ne y$. Let $r = \frac{1}{2}|x-y| > 0$ to show that $B_r(x,y) \subset L^c$. To do this, choose $(u,v) \in B_r(x,y)$ so $\|(x,y) - (u,v)\| < r$ to show $(u,v) \notin L$, or in other words, $u \ne v$. Using the reverse triangle inequality

$$\begin{aligned} |u - v| &= |(x-y) - (x-u) - (v-y)| \ge |x-y| - |x-u| - |v-y| \\ &\ge |x-y| - 2\|(x-u, y-v)\| > r - 2\left(\frac{1}{2}r\right) = 0. \end{aligned}$$

   Thus all points of $B_r(x,y)$ miss L, thus $B_r(x,y) \subset L^c$, so L^c is open and L is closed.

2. *Let $\{\mathbf{x}_n\} \in \mathbf{R}^d$ be a sequence and $\mathbf{x}, \mathbf{v} \in \mathbf{R}^d$. Define: $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. Recall that the one-norm $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_d|$ is the sum of the absolute values of the components of $\mathbf{x} = (x_1, \ldots, x_d)$. Suppose $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. Show that*

$$z_n = \mathbf{x}_n \cdot \mathbf{v} + \|\mathbf{x}_n - \mathbf{x}\|_1 \to \mathbf{x} \cdot \mathbf{v} \qquad \text{as } n \to \infty.$$

   $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$ means that for all $\epsilon > 0$ there is $N \in \mathbf{R}$ such that

$$\|\mathbf{x}_n - \mathbf{x}\| < \epsilon \qquad \text{whenever } n > N,$$

   where $\|\mathbf{x}\|$ denotes the Euclidean norm.

   The one-norm is bounded by the Euclidean norm since for all $\mathbf{y}$,

$$\|\mathbf{y}\|_1 = |y_1| + \cdots |y_d| \le \|\mathbf{y}\| + \cdots + \|\mathbf{y}\| = d\|\mathbf{y}\|.$$

   To show that $\{z_n\}$ converges, choose $\epsilon > 0$. By the convergence of $\{\mathbf{x}_n\}$, there is $N \in \mathbf{R}$ such that

$$\|\mathbf{x}_n - \mathbf{x}\| < \frac{\epsilon}{\|\mathbf{v}\| + d} \qquad \text{whenever } n > N,$$

   For $n > N$ and using the Schwarz inequality,

$$\begin{aligned} |z_n - \mathbf{x} \cdot \mathbf{v}| &= |\mathbf{x}_n \cdot \mathbf{v} + \|x_n - x\|_1 - \mathbf{x} \cdot \mathbf{v}| \\ &= |(\mathbf{x}_n - \mathbf{x}) \cdot \mathbf{v} + \|\mathbf{x}_n - \mathbf{x}\|_1| \\ &\le |(\mathbf{x}_n - \mathbf{x}) \cdot \mathbf{v}| + \|\mathbf{x}_n - \mathbf{x}\|_1 \\ &\le \|\mathbf{x}_n - \mathbf{x}\| \, \|\mathbf{v}\| + d\|\mathbf{x}_n - \mathbf{x}\| \\ &\le \|\mathbf{x}_n - \mathbf{x}\| \left(\|\mathbf{v}\| + d\right) \\ &< \frac{\epsilon}{\|\mathbf{v}\| + d} \left(\|\mathbf{v}\| + d\right) = \epsilon. \end{aligned}$$

   Thus $z_n \to \mathbf{x} \cdot \mathbf{v}$ as $n \to \infty$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Let $\mathcal{O}_i \subset \mathbf{R}^2$ be an open set for every $i \in \mathbb{N}$. Then the intersection $\bigcap_{i\in\mathbb{N}} \mathcal{O}_i$ is open.*

   FALSE. Let $\mathcal{O}_i = B_{1/i}(0)$ be the balls about the origin of radius $\frac{1}{i}$ which are open in $\mathbf{R}^2$. Then $\bigcap_{i\in\mathbb{N}} \mathcal{O}_i = \{(0,0)\}$, a single point, which is not open.

   (b) STATEMENT: *Let $E \subset \mathbf{R}^d$. If every sequence $\{\mathbf{x}_n\} \subset E$ has a subsequence that converges to a point of E, then E is compact.*

   TRUE. We show that E is closed and bounded, therefore compact by the Heine Borel Theorem. To see that E is closed, we only need to check that it contains the limit points of its convergent subsequences. But by assumption, such sequences have convergent subsequences that converge to a point of E. However, the subsequential limit of a convergent sequence is the limit, thus is a point of E. Hence, E is closed.

   To see E is bounded, argue by contrapositive. If E were not bounded, then choose a sequence as follows: let $\mathbf{x}_1 \in E$ be any. By unboundedness, choose $\mathbf{x}_2 \in E$ such that $\|\mathbf{x}_2\| > \|\mathbf{x}_1\| + 2$. Continue the same way: if $\mathbf{x}_1, \ldots, \mathbf{x}_k$ have been chosen, choose $\mathbf{x}_{k+1} \in E$ such that $\|\mathbf{x}_{k+1}\| \ge k + 1 + \|\mathbf{x}_k\|$. The resulting sequence has no convergent subsequence since every subsequence is unbounded: the norms of terms in every subsequence tend to infinity. Thus the condition fails: there is a sequence that does not have any convergent subsequence.

   This condition on E is called *sequential compactness.*

   (c) *Let $E \subset \mathbf{R}^d$. If both the interior E° and the boundary ∂E are connected, then E is connected.*

   FALSE. Consider the set

   $$E = \{(x, y) \in \mathbf{R}^2 : 0 \le x \le 2,\ 0 \le y \le 1 \text{ and } x \text{ is rational if } x < 1\}.$$

   Then $\partial E = [0,1] \times [0,1] \cup \{(x,y) \in (1,2] \times [0,1] : x = 2 \text{ or } y = 0 \text{ or } y = 1\}$ and $E^\circ = (1,2) \times (0,1)$ are both connected but E is not: $U = (-\infty, r) \times \mathbf{R}$ and $V = (r, \infty) \times \mathbf{R}$ is a separation of E, where $r \in (0,1)$ is any irrational.

4. *Let (X, d) be a metric space and $K \subset X$. Define: the set K is* compact. *Let K and C be subsets of the metric space X. Using just your definition, show that if K is compact and C is closed then $K \cap C$ is compact.*

   K is compact if every open cover has a finite subcover. That is, if $\mathcal{O}_\alpha$ are open sets for all $\alpha \in A$ such that $K \subset \bigcup_{\alpha\in A} \mathcal{O}_\alpha$, then there are finitely many $\alpha_1, \ldots, \alpha_\ell$ such that

   $$K \subset \mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_\ell}.$$

   To show $K \cup C$ is compact, take any open cover $\{\mathcal{O}_\alpha\}_{\alpha\in A}$. By augmenting the cover by another open set C^c, the complement of C, we have a cover $K \subset \left(\bigcup_{\alpha\in A} \mathcal{O}_\alpha\right) \cup C^c$. By compactness, there are finitely many $\alpha_1, \ldots, \alpha_\ell$ such that

   $$K \subset \mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_\ell} \cup C^c,$$

   where we've thrown in C^c whether we need it or not. It follows that the smaller set

   $$K \cap C \subset \mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_\ell}.$$

   We have shown that any cover has a finite subcover, thus $K \cap C$ is compact.

5. *Let $E, F \subset \mathbf{R}^d$. Define: $\overline{E}$, the closure of E. State a theorem that gives a condition for $x \in \mathbf{R}^d$ to be a point of $\overline{E}$. Prove that $\overline{E \cup F} = \left(\overline{E}\right) \cup \left(\overline{F}\right)$.*

The closure of E is the smallest closed set that contains E. Using the fact that intersections of closed sets are closed, this may be written

$$\overline{E} = \bigcap \{F : F \text{ is closed and } E \subset F\}$$

Theorem. Let $E \subset \mathbf{R}^d$ and $\mathbf{x} \in \mathbf{R}^d$. Then $\mathbf{x} \in \overline{E}$ if and only if for all $r > 0$ we have $B_r(\mathbf{x}) \cap E \neq \emptyset$.

To show $\overline{E \cup F} = \overline{E}\ \cup \overline{F}$ we show $\overline{E \cup F} \subset \overline{E} \cup \overline{F}$ and $\overline{E \cup F} \supset \overline{E} \cup \overline{F}$.

To show $\overline{E \cup F} \subset \overline{E}\ \cup \overline{F}$ choose $\mathbf{x} \in \overline{E \cup F}$. Hence for all $n \in \mathbf{N}$ we have $B_{1/n}(\mathbf{x}) \cap (E \cup F) \neq \emptyset$ by the theorem. Thus, for each n either $B_{1/n}(\mathbf{x}) \cap E \neq \emptyset$ or $B_{1/n}(\mathbf{x}) F \neq \emptyset$. Thus infinitely many n's meet one set or the other, otherwise $B_{1/n}(\mathbf{x}) \cap (E \cup F) = \emptyset$ for large n. Say that E is met infinitely often, *i.e.,* there is a subsequence n_j increasing to infinity such that $B_{1/n_j}(\mathbf{x}) \cap E \neq \emptyset$ for all j. But this implies $B_r(\mathbf{x}) \cap E \neq \emptyset$ for all $r > 0$. This is because, for any $r > 0$ there is j such that $r > 1/n_j$ and so $B_r(\mathbf{x}) \cap E \supset B_{1/n_j}(\mathbf{x}) \cap E \neq \emptyset$. It follows that $\mathbf{x} \in \overline{E}$, thus $\mathbf{x} \in \overline{E} \cup \overline{F}$. If F was met infinitely often instead, swap the roles of E and F in the remainder of the argument.

To show $\overline{E} \cup \overline{F} \subset \overline{E \cup F}$ choose $\mathbf{x} \in \overline{E} \cup \overline{F}$. Hence $\mathbf{x} \in \overline{E}$ or $\mathbf{x} \in \overline{F}$. Say $\mathbf{x} \in \overline{E}$. Then for all $r > 0$ we have $B_r(\mathbf{x}) \cap E \neq \emptyset$. It follows that for all $r > 0$ we have $B_r(\mathbf{x}) \cap (E \cup F) \neq \emptyset$ which is a larger set. Thus $\mathbf{x} \in \overline{E \cup F}$. A similar argument holds in case $\mathbf{x} \in \overline{F}$ instead.

In contrast, note that $\overline{E \cap F} = \overline{E} \cap \overline{F}$ fails sometimes. For example, if $E = (0, 1)$ and $F = (1, 2)$ in $\mathbf{R}$, then $\overline{E \cap F} = \emptyset$ but $\overline{E} \cap \overline{F} = \{1\}$.