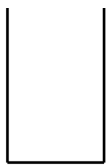
Math 3220 § 2. **Third Midterm Exam** Name: Solutions
Treibergs November 20, 2019

1. *Let $\mathcal{U} \subset \mathbb{R}^2$ be the set consisting of the line segments from $(0,8)$ to $(0,0)$ to $(5,0)$ to $(5,8)$ pictured below. Determine whether $\mathcal{U}$ is a Jordan Region and find its upper and lower volumes. Explain. What is the lower volume $\underline{V}(\mathcal{U})$? What is the upper volume $\overline{V}(\mathcal{U})$?*



The set is a region of volume zero, hence a Jordan region whose upper and lower volumes are zero $\underline{V}(\mathcal{U}) = \overline{V}(\mathcal{U}) = 0$. To see this, we shall use the Theorem about zero volume sets: a bounded set $\mathcal{U} \subset \mathbb{R}^2$ has volume zero if and only if for every $\epsilon > 0$ there is a finite cover of $\mathcal{U}$ by aligned rectangles of total volume less than ϵ. Choose $\epsilon > 0$. Take three sets that cover the three sides of $\mathcal{U}$

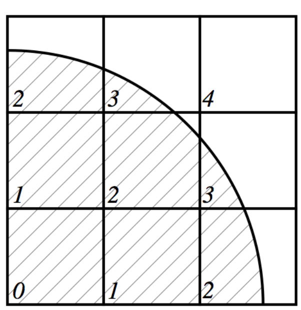
$$R_1 = [-\epsilon, \epsilon] \times [-\epsilon, 8+\epsilon]; \qquad R_2 = [-\epsilon, 5+\epsilon] \times [-\epsilon, \epsilon], \qquad R_3 = [5-\epsilon, 5+\epsilon] \times [-\epsilon, 8+\epsilon].$$

These sets cover: $\mathcal{U} \subset R_1 \cup R_2 \cup R_3$. But the total volume is

$$V(R_1) + V(R_2) + V(R_3) = 2\epsilon(8+2\epsilon) + (5+2\epsilon)2\epsilon + 2\epsilon(8+2\epsilon) = 42\epsilon + 12\epsilon^2$$

which can be made arbitrarily small since we may choose ϵ as small as we please. Hence $\mathcal{U}$ has volume zero.

2. *Let $R \subset \mathbb{R}^d$ be an aligned rectangle, $g : R \to \mathbb{R}$ be a bounded function and $A \subset R$ be a subset. State the definition: g is integrable on R. State the definition: g is integrable on A. Let $S = [0,3] \times [0,3]$, $B = \{(x,y) \in S : x^2 + y^2 \le 7\}$ and $f(x,y) = \lfloor x \rfloor + \lfloor y \rfloor$, where the "floor" (or greatest integer) function is given by $\lfloor x \rfloor = n$ where n is an integer such that $n \le x < n+1$. The values of f on the interiors and the left and bottom edges of the unit squares are given in the lower left corners of the squares. Using the theorems from the text, determine whether this f is integrable on B. (You do not need to integrate.) For the partition $\mathcal{P} = \{\{0 < 1 < 2 < 3\}, \{0 < 1 < 2 < 3\}\}$ of S, determine the lower and upper sums for f on B. What does this say about $\int_B f\,dV$ assuming f is integrable on B?*



The bounded function g is *integrable* on R if lower integral equals the upper integral

$$\underline{\int}_R g\,dV = \overline{\int}_R g\,dV,$$

where

$$\underline{\int}_R g\,dV = \sup_{\mathcal{P}} L(g, \mathcal{P}), \qquad \overline{\int}_R g\,dV = \inf_{\mathcal{P}} U(g, \mathcal{P}),$$

where sup and inf are taken over partitions $\mathcal{P}$ of R .

g is *integrable* on A if for any aligned rectangle R containing A, the function $g_A = \chi_A g$ is integrable on R, where χ_A is the characteristic function of A. In other words

$$g_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

The region B is defined by

$$B = \left\{ (x, y) \in \mathbb{R}^2 : 0 \le x \le \sqrt{7}, 0 \le y \le \sqrt{7 - x^2} \right\}.$$

This is a closed and bounded region, hence compact. Furthermore, because the upper and lower functions $\psi(x) = 0$ and $\phi(x) = \sqrt{7 - x^2}$ are continuous on $[0, \sqrt{7}]$, the region B is a Jordan Region, according to our homework problem. Indeed, one can check that $V(\partial B) = 0$. The function is piecewise continuous. The discontinuity set is

$$E = \{(x, y) \in R : f \text{ is not continuous at } (x, y)\}$$

which is contained in the set of lines at the integers

$$\begin{aligned} L = \{(x, y) \in \mathbb{R}^2 &: x \in \{0, 1, 2, 3\} \text{ and } 0 \le y \le 3\} \\ \cup \{(x, y) \in \mathbb{R}^2 &: 0 \le x \le 3 \text{ and } y \in \{0, 1, 2, 3\} \ \} \end{aligned}$$

which has volume zero. But a function defined on a Jordan Region, whose discontinuities are contained in a set of volume zero is integrable, by a theorem in the text and covered in class.

Finally, we compute the upper lower sum and upper sum for $f_B = \chi_B f$. Letting m_{ij} denote the inf and M_{ij} denote the sup in each of the unit squares, we see that

$$m_{ij} = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array} \qquad M_{ij} = \begin{array}{|c|c|c|} \hline 3 & 3 & 0 \\ \hline 3 & 3 & 3 \\ \hline 2 & 3 & 3 \\ \hline \end{array}$$

For example in the lower left closed square, the function is zero except on the $x = 1$ and $y = 1$ edges where $f = 1$ and $f(1, 1) = 2$ at the corner. Thus $m_{11} = 0$ and $M_{11} = 2$. Thus the lower and upper sums are

$$\begin{aligned} L(f_B, \mathcal{P}) &= \sum m_{ij} V(R_{ij}) = (0 + 1 + 0 + 1 + 0 + 0 + 0 + 0 + 0) \cdot 1 = 2, \\ L(f_B, \mathcal{P}) &= \sum M_{ij} V(R_{ij}) = (2 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 0) \cdot 1 = 23. \end{aligned}$$

It follows that

$$2 = L(f_B, \mathcal{P}) \le \int_R f_B \, dV = \int_B f \, dv \le U(f_B, \mathcal{P}) = 23.$$

3. (a) *Complete the statement of the following theorem:*

Theorem. *Let $R \subset \mathbb{R}^2$ be an aligned rectangle and $f : R \to \mathbb{R}$ be a bounded function. Then f is* integrable *on R if and only if*

$$\boxed{\text{for every } \epsilon > \text{ there is a partition } \mathcal{P} \text{ of } R \text{ such that } U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.}$$

(b) *Using just your theorem , prove that $f(x,y)$ is integrable on $R = [0,8] \times [0,6]$ where*

$$f(x,y) = \begin{cases} 1, & \text{if } 3 < x < 6 \text{ and } 2 < y < 4; \\ 0, & \text{otherwise.} \end{cases}$$

Choose $\epsilon > 0$. Consider the partition $\mathcal{P} = \{\{0, 3, 3+\epsilon, 6-\epsilon, 6, 8\}, \{0, 2, 2+\epsilon, 4-\epsilon, 4, 6\}\}$. Then $M_{ij} = m_{ij} = 0$ for the sixteen outside subrectangles; $M_{ij} = 1$ and $m_{ij} = 0$ for the next concentric eight subrectangles along the boundary of $[3,6] \times [2,4]$ and $M_{ij} = m_{ij} = 1$ for the center subrectangle. Thus, only the eight touching the boundary of $(3,6) \times (2,4)$ survive in

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{ij} (M_{ij} - M_{ij})\, V(R_{ij}) \\ &= (1-0)\big(V(R_{22}) + V(R_{23}) + V(R_{24}) + V(R_{32}) + \\ &\qquad + V(R_{34}) + V(R_{42}) + V(R_{43}) + V(R_{44})\big) \\ &= \epsilon^2 + (3-2\epsilon)\epsilon + \epsilon^2 + \epsilon(2-2\epsilon) + \epsilon(2-2\epsilon) + \epsilon^2 + (3-2\epsilon)\epsilon + \epsilon^2 \\ &= 10\epsilon - 4\epsilon^2. \end{aligned}$$

This may be made as small as we please since $\epsilon > 0$ was arbitrary. Hence by the Theorem, f is integrable on R.

4. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a) STATEMENT. *Let $E \subset \mathbf{R}^2$ be a bounded set and $f : E \to \mathbf{R}$ be a bounded function. If the lower integral $\underline{\int}_E f(x)\, dv(x) > 0$ then f is integrable on E.*

FALSE. For $E = [0,1] \times [0,1]$, we find a function with positive lower integral which does not equal its upper integral, hence is not integrable on E. Indeed, for

$$f(x,y) = \begin{cases} 2, & \text{if } y \in \mathbb{Q}; \\ 1, & \text{otherwise.} \end{cases}$$

we have

$$0 < \underline{\int}_E f(x,y)\, dV(x,y) = 1 < \overline{\int}_E f(x,y)\, dV(x,y) = 2.$$

(b) Statement. *If the bounded function $f : [0,1]\times[0,2] \to \mathbf{R}$ is integrable on $[0,1]\times[0,2]$ then $f(x,y)$ is an integrable function of y on $[0,2]$ for every $x \in [0,1]$.*

False. Let

$$f(x,y) = \begin{cases} 1, & \text{if } x = \frac{1}{2} \text{ and } q \in \mathbb{Q}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $f(x,y)$ is discontinuous on the set $\left\{\frac{1}{2}\right\} \times [0,2]$ in $R = [0,1]\times[0,2]$ which has volume zero. Hence $f(x,y)$ is integrable on R, but the function $f\left(\frac{1}{2}, y\right)$ is not integrable for $y \in [0,2]$.

(c) Statement. *Let $f_n(x,y) : [0,1]\times[0,1] \to \mathbf{R}$ be integrable. If the sequence of $f_n \to 0$ pointwise on $[0,1]\times[0,1]$ as $n \to \infty$, then* $\displaystyle\lim_{n\to\infty} \int_{[0,1]\times[0,1]} f_n(x,y)\, dV(x,y) = 0.$

False. Consider the sequence of functions

$$f_n(x,y) = \begin{cases} n^2, & \text{if } (x,y) \in \left(0, \frac{1}{n}\right) \times \left(0, \frac{1}{n}\right); \\ 0, & \text{otherwise.} \end{cases}$$

Then for every $(x,y) \in R = [0,1]\times[0,1]$ we have $f_n(x,y) \to 0$ as $n \to \infty$. To see it, if $(x,y) \in \partial R$ then $f_n(x,y) = 0$ for all n. Otherwise, for n large enough so that $\frac{1}{n} < \min\{x,y\}$ we have $f_n(x,y) = 0$, hence $f_n(x,y) \to 0$ as $n \to \infty$ also. On the other hand, the bounded function f_n is integrable on R since its discontinuities are a set of volume zero but its integral is for all n,

$$\int_R f_n(x,y)\, dV(x,y) = 1.$$

5. *Let $F(x,y,z,w) = (x + yz + w,\ xy + zw)$. Prove that the level set*

$$\mathcal{S} = \{(x,y,z,w) : F(x,y,z,w) = (5,4)\}$$

is a locally parameterized surface near the point $(1,2,1,2) \in \mathcal{S}$. What is the tangent space to $\mathcal{S}$ at $(1,2,1,2)$?

We apply the Implicit Function Theorem to the smooth function F. Compute the differential

$$dF(x,y,z,w) = \begin{pmatrix} 1 & z & y & 1 \\ y & x & w & z \end{pmatrix}.$$

At the point $P = (1,2,1,2)$,

$$dF(P) = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}.$$

To solve for (x,y) in terms of (z,w) we need that the $dF_{x,y}(P)$ part of the differential to be invertible. Indeed

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

has determinant -1 so is invertible. By the Implicit Function Theorem, there is an open set $\mathcal{G} \subset \mathbb{R}^4$ such that $(1,2,1,2) \in \mathcal{G}$ and an open set $\mathcal{H} \subset \mathbb{R}^2$ such that $(1,2) \in \mathcal{H}$ and $\mathcal{C}^1$ functions $u, v : \mathcal{H} \to \mathbb{R}$ such that $u(1,2) = 1$, $v(1,2) = 2$ and

$$F(x,y,z,w) = (5,4) \text{ for } (x,y,z,w) \in \mathcal{G} \quad \Longleftrightarrow \quad \begin{array}{l} x = u(z,w) \text{ and } y = v(z,w) \\ \text{for some } (z,w) \in \mathcal{H}. \end{array}$$

The level set $\mathcal{S}$ is represented as a parameterized surface by the parameterization $\Phi : \mathcal{H} \to \mathbb{R}^4$ given by $\Phi(z,w) = (u(z,w), v(z,w), z, w)$. We have

$$\mathcal{S} \cap \mathcal{G} = \{\Phi(z,w) : (z,w) \in \mathcal{H}\}.$$

The tangent plane is given by $\Phi(1,2) + d\Phi(1,2)(\mathbb{R}^2)$. But, by the chain rule, differentiating the equation for $\mathcal{S}$, namely $dF(\Phi(z,w)) = (5,4)$, we get $dF(P) \circ d\Phi(1,2)(\mathbb{R}^2) = 0$. Thus the tangent plane is the affine space that passes through $P = (1,2,1,2)$ and is parallel to the nullspace of $dF(P)$. Solving using row operations on the matrix we find

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ \\ 2 & 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ \\ 0 & -1 & -2 & -1 \end{pmatrix}$$

so that

$$\begin{aligned} y &= -2z - w \\ x &= -y - 2z - w = 0. \end{aligned}$$

Thus the tangent space to $\mathcal{S}$ at P is

$$\left\{ \begin{pmatrix} 1 \\ \\ 2 - 2z - w \\ \\ 1 + z \\ \\ 2 + w \end{pmatrix} : z, w \in \mathbb{R} \right\}.$$