Math 3220 § 2. **Second Midterm Exam** Name: Solutions
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1. *Let* $f(x,y) = \begin{cases} \dfrac{y^5}{x^4+y^4}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

Is f is continuous at $(0,0)$? Do the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$? Is f differentiable at $(0,0)$? Why?

The function IS CONTINUOUS at $(0,0)$. Indeed

$$|f(h,k) - f(0,0)| = \left| \frac{k^5}{h^4+k^4} - 0 \right| = \frac{|k|k^4}{h^4+k^4} \le |k| \le \sqrt{h^2+k^2} = \|(h,k)-(0,0)\| \to 0$$

as $(h,k) \to (0,0)$.

The PARTIAL DERIVATIVES EXIST at $(0,0)$. We have $f(x,0) = 0$ and $f(0,y) = \dfrac{y^5}{0^4+y^4} = y$. It follows that

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{h\to 0} \frac{k-0}{k} = 1.$$

$f(x,y)$ is NOT DIFFERENTIABLE at $(0,0)$. If the differential exsted, it would be the linear transformation involving the Jacobian matrix

$$df(0,0)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x}(0,0) & \dfrac{\partial f}{\partial y}(0,0) \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = k.$$

But the difference quotient does not limit to zero. The difference quotient is

$$q(h,k) = \frac{f(h+0,k+0) - f(0,0) - df(0,0)(h,k)}{\|(h,k)\|} = \frac{\dfrac{k^5}{h^4+k^4} - 0 - k}{\sqrt{h^2+k^2}}$$
$$= \frac{k^5 - k(h^4+k^4)}{(h^4+k^4)\sqrt{h^2+k^2}} = -\frac{kh^4}{(h^4+k^4)\sqrt{h^2+k^2}}.$$

Along the path $(h,k) = (t,0)$ we have $q(t,0) = 0 \to 0$ as $t \to 0$. But along another path $(h,k) = (t,t)$,

$$q(t,t) = -\frac{t^5}{(t^4+t^4)\sqrt{t^2+t^2}} = -\frac{1}{2\sqrt{2}} \to -\frac{1}{2\sqrt{2}}$$

as $t \to 0$. Thus the limits along two paths are unequal so the limit $\lim_{(h,k)\to(0,0)} q(h,k)$ does not even exist.

2. *Let $f(x,y) : \mathbf{R}^2 \to \mathbf{R}$ be a function whose first partial derivatives exist everywhere. For $(a,b) \in \mathbf{R}^2$, let*

$$g(x,y) = \begin{cases} \dfrac{f(x,y) - f(a,y) - f(x,b) + f(a,b)}{(x-a)(y-b)}, & \text{if } (x,y) \neq (a,b); \\ c, & \text{if } (x,y) = (a,b). \end{cases}$$

Assume that $g(x,y)$ is continuous at (a,b). Deduce that both $\dfrac{\partial^2 f}{\partial x\, \partial y}$ and $\dfrac{\partial^2 f}{\partial y\, \partial x}$ exist and are equal at (a,b).

Strictly speaking, the function $g(x,y)$ is not defined if $x = a$ or $y = b$. However, using the assumption that partial derivatives exist, the limits exist at $x = a$ and $y \neq b$

$$\lim_{x \to a} g(x,y) = \frac{1}{y-b} \left(\frac{\partial f}{\partial x}(a,y) - \frac{\partial f}{\partial x}(a,b) \right)$$

and for $y = b$ and $x \neq a$

$$\lim_{y \to b} g(x,y) = \frac{1}{x-a} \left(\frac{\partial f}{\partial y}(x,b) - \frac{\partial f}{\partial y}(a,b) \right).$$

Thus we may extend the function to $(x,y) \neq (a,b)$ by

$$\tilde{g}(x,y) = \begin{cases} \dfrac{f(x,y) - f(a,y) - f(x,b) + f(a,b)}{(x-a)(y-b)}, & \text{if } x \neq a \text{ and } y \neq b; \\ \dfrac{1}{y-b} \left(\dfrac{\partial f}{\partial x}(a,y) - \dfrac{\partial f}{\partial x}(a,b) \right), & \text{if } x = a \text{ and } y \neq b; \\ \dfrac{1}{x-a} \left(\dfrac{\partial f}{\partial y}(x,b) - \dfrac{\partial f}{\partial y}(a,b) \right), & \text{if } y = b \text{ and } x \neq a; \\ c, & \text{if } (x,y) = (a,b). \end{cases}$$

This function also satisfies

$$c = \tilde{g}(a,b) = \lim_{(x,y) \to (a,b)} \tilde{g}(h,k).$$

The two second derivatives are the limits as $(h,k) \to (0,0)$ for different paths, hence are equal to the limit. By continuity of $\tilde{g}$, the limit exists and equals the value at (a,b). By going $x \to a$ first and then $y \to b$ we have

$$\begin{aligned} c = \tilde{g}(a,b) &= \lim_{(x,y) \to (a,b)} \tilde{g}(h,k) \\ &= \lim_{y \to b} \frac{1}{y-b} \left(\lim_{x \to a} \frac{f(x,y) - f(a,y) - f(x,b) + f(a,b)}{x-a} \right) \\ &= \lim_{y \to b} \frac{1}{y-b} \left(\frac{\partial f}{\partial x}(a,y) - \frac{\partial f}{\partial x}(a,b) \right) \\ &= \frac{\partial^2 f}{\partial y\, \partial x}(0,0) \end{aligned}$$

where in the inner limit we used the assumption that $\dfrac{\partial f}{\partial x}(x,y)$ exists for all (x,y). Thus the partial derivative $\dfrac{\partial^2 f}{\partial y\, \partial x}(0,0)$ exists at (a,b) and equals c. Instead, by going $y \to b$ first and then $x \to a$ we have

$$\begin{aligned}
c = g(a,b) &= \lim_{(x,y)\to(a,b)} g(h,k) \\
&= \lim_{x\to a} \frac{1}{x-a}\left(\lim_{y\to b} \frac{f(x,y)-f(x,b)-f(a,y)+f(a,b)}{y-b}\right) \\
&= \lim_{x\to a} \frac{1}{x-a}\left(\frac{\partial f}{\partial y}(x,b) - \frac{\partial f}{\partial y}(a,b)\right) \\
&= \frac{\partial^2 f}{\partial x\,\partial y}(0,0)
\end{aligned}$$

where in the inner limit we used the other assumption that $\dfrac{\partial f}{\partial y}(x,y)$ exists for all (x,y). Thus the partial derivative $\dfrac{\partial^2 f}{\partial x\,\partial y}(0,0)$ exists at (a,b) and also equals c.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Let $D \subset \mathbf{R}$ and $\gamma : D \to \mathbf{R}^2$ be continuous. If $\gamma(D)$ is connected then D is connected.*

   FALSE. The set $D = (0,1) \cup (2,3)$ is disconnected but for $\gamma(t) = (0,0)$, constant, the singleton set $\gamma(D) = \{(0,0)\}$ is connected.

   (b) STATEMENT: *Let $E \subset \mathbf{R}^p$ and $f : E \to \mathbf{R}^q$ be continuous. If A is relatively closed in $f(E)$ then $f^{-1}(A)$ is relatively closed in E.*

   TRUE. A being relatively closed in $f(E)$ means that $f(E)\backslash A$ is relatively open in $f(E)$. Hence there is an open set $U \subset \mathbf{R}^q$ so that $f(E)\backslash A = U \cap f(E)$. It follows that

$$\begin{aligned}
E\backslash f^{-1}(A) &= f^{-1}(f(E)\backslash A) = f^{-1}(U \cap f(E)) \\
&= f^{-1}(U) \cap f^{-1}(f(E)) = f^{-1}(U) \cap E = f^{-1}(U),
\end{aligned}$$

   which is reltively open in E because f is continuous. It follows that $f^{-1}(A)$ is relatively closed in E.

   (c) STATEMENT: $f(x,y) \sim \displaystyle\sum_{k=0}^{\infty} \left(x^2+y^2\right)^k$ *converges uniformly in the unit ball $B_1(0,0)$.*

   FALSE. The partial sums $f_n(x,y) = \displaystyle\sum_{k=0}^{n} \left(x^2+y^2\right)^k$ are bounded for $(x,y) \in B_1(0,0)$,

$$|f_n(x,y)| \le \sum_{k=0}^{n} \left|x^2+y^2\right|^k \le \sum_{k=0}^{n} 1 = n+1$$

   since $x^2+y^2 < 1$. If the convergence were uniform, $\{f_n\}$ would limit to a bounded function, contradicting the fact that the sum is unbounded on $B_1(0,0)$

$$f(x,y) = \sum_{k=0}^{\infty} \left(x^2+y^2\right)^k = \frac{1}{1-x^2-y^2}.$$

   We have used the formula for the sum of a geometric series $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ with $r = x^2+y^2$.

Other proofs are possible based on other properties of uniformly convergent series. For example, if $f_n \to f$ is uniform as $n \to \infty$ then $f_n(x_n, y_n) - f(x_n, y_n) \to 0$ for any sequence $\{(x_n, y_n)\}$ in $B_1(0,0)$. However, choosing $(x_n, y_n) = \left(\sqrt{1 - \frac{1}{n^2}}, 0\right)$ we have $f_n(x_n, y_n) - f(x_n, y_n) \le n + 1 - n^2 \to -\infty$ as $n \to \infty$, so the convergence couldn't have been uniform.

4. *Let $K \subset \mathbf{R}^p$. Define: the set K is compact. Suppose that $\mathbf{f} : \mathbf{R}^p \to \mathbf{R}^q$ is a $\mathcal{C}^1$ function. Show that for every $r > 0$ there is an $M \in \mathbf{R}$ such that for all $\mathbf{x}, \mathbf{y} \in B_r(\mathbf{0})$, the ball of radius r,*

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \le M\|\mathbf{x} - \mathbf{y}\|.$$

A set $K \subset \mathbf{R}^p$ is *compact* if every open cover has an finite subcover, *i.e.*, if $K \subset \cup_{\alpha \in A} G_\alpha$ for some open sets G_α, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ so that $K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$.

Writing the function in components, $\mathbf{f}(\mathbf{x}) = (f_1(x_1, \ldots, x_p), \ldots, f_q(x_1, \ldots, x_p))$, its differential is the linear transformation given by the $q \times p$ Jacobian matrix

$$d\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_p}(x) \\ \vdots & & \vdots \\ \frac{\partial f_q}{\partial x_1}(x) & \cdots & \frac{\partial f_q}{\partial x_p}(x) \end{pmatrix}.$$

Because $\mathbf{f} \in \mathcal{C}^1(\mathbf{R}^p, \mathbf{R}^q)$, the partial derivatives are all continuous. Hence the function

$$M(\mathbf{x}) = \sqrt{\sum_{i=1}^{p} \sum_{j=1}^{q} \left(\frac{\partial f_j}{\partial x_i}(\mathbf{x})\right)^2}$$

is continuous. It is also a bound on the operator norm of the differential: for all $\mathbf{x}, \mathbf{h} \in \mathbf{R}^p$

$$\|d\mathbf{f}(\mathbf{x})(\mathbf{h})\| \le M(\mathbf{x})\|\mathbf{h}\|. \tag{1}$$

Choose $r > 0$. The closed ball $\overline{B_r(\mathbf{0})}$ is compact so that the continuous function has a bound on $\overline{B_r(\mathbf{0})}$ given by

$$M_r = \sup_{\mathbf{x} \in \overline{B_r(\mathbf{0})}} M(\mathbf{x}). \tag{2}$$

Now choose any $\mathbf{x}, \mathbf{y} \in B_r(\mathbf{0})$. Since $B_r(\mathbf{0})$ is convex, the line segment $[\mathbf{x}, \mathbf{y}] \subset B_r(\mathbf{0})$. It follows from the Mean Value Theorem that there is a point $\mathbf{z} \in [\mathbf{x}, \mathbf{y}] \subset \overline{B_r(\mathbf{0})}$ so that

$$\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = df(\mathbf{z})(\mathbf{y} - \mathbf{x}).$$

By taking norms and using (1) and (2),

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})\| = \|d\mathbf{f}(\mathbf{z})(\mathbf{y} - \mathbf{x})\| \le M(\mathbf{z})\|\mathbf{y} - \mathbf{x}\| \le M_r\|\mathbf{y} - \mathbf{x}\|.$$

5. *Let $\mathbf{a}, \mathbf{h} \in \mathbf{R}^2$, $f : \mathbf{R}^3 \to \mathbf{R}^2$ and $g : \mathbf{R}^2 \to \mathbf{R}^3$ be given by*

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \mathbf{h} = \begin{pmatrix} k \\ \ell \end{pmatrix}, \qquad f\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2u - v \\ vw \end{pmatrix}, \qquad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ xy \\ -1 \end{pmatrix}.$$

Compute $d(f\circ g)(\mathbf{a})(\mathbf{h})$ in two ways, directly and using the chin rule.

The first way is to compute $f\circ g$ and take its differetial.

$$f\circ g\begin{pmatrix}x\\y\end{pmatrix}=f\left(g\begin{pmatrix}x\\y\end{pmatrix}\right)=f\begin{pmatrix}x+y\\xy\\-1\end{pmatrix}=\begin{pmatrix}2(x+y)-xy\\-xy\end{pmatrix},$$

$$d(f\circ g)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2-y & 2-x\\-y & -x\end{pmatrix},$$

$$d(f\circ g)(\mathbf{a})(\mathbf{h})=d(f\circ g)\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}k\\\ell\end{pmatrix}=\begin{pmatrix}0 & 1\\-2 & -1\end{pmatrix}\begin{pmatrix}k\\\ell\end{pmatrix}.$$

The second way is to use the chain rule: take differentials first and multiply.

$$\mathbf{b}=g(\mathbf{a})=g\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\2\\-1\end{pmatrix},$$

$$dg\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1 & 1\\y & x\\0 & 0\end{pmatrix}\quad\text{so}\quad dg(\mathbf{a})=dg\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1 & 1\\2 & 1\\0 & 0\end{pmatrix}$$

$$df\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}2 & -1 & 0\\0 & w & v\end{pmatrix}\quad\text{so}\quad df(\mathbf{b})=df\begin{pmatrix}3\\2\\-1\end{pmatrix}=\begin{pmatrix}2 & -1 & 0\\0 & -1 & 2\end{pmatrix}$$

By the chain rule,

$$\begin{aligned}d(f\circ g)(\mathbf{a})(\mathbf{h})&=df\big(g(\mathbf{a})\big)\big(df(\mathbf{a})(\mathbf{h})\big)=df(\mathbf{b})\,dg(\mathbf{a})(\mathbf{h})\\&=\begin{pmatrix}2 & -1 & 0\\0 & -1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}k\\\ell\end{pmatrix}=\begin{pmatrix}0 & 1\\-2 & -1\end{pmatrix}\begin{pmatrix}k\\\ell\end{pmatrix}.\end{aligned}$$