Math 3220 § 2. **First Midterm Exam** Name:= Solutions

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1. *Let* $E, F, G \subset \mathbf{R}^d$. *Define: G is an* open set. *Define: F is a* closed set. *Using just your definitions in (a), show that if both E and F are closed then their union* $E \cup F$ *is closed.*

   A set $G \subset \mathbf{R}^d$ is *open* if for every $x \in G$ there is an $\epsilon > 0$ so that the ball $B_\epsilon(x) \subset G$. A set $F \subset \mathbf{R}^d$ is *closed* if its complement $F^c = \mathbf{R}^d \backslash F$ is open.

   To show $E \cup F$ is closed we have to show that its complement

   $$(E \cup F)^c = E^c \cap F^c$$

   is open, where we have used deMorgan's law. Let $x \in E^c \cap F^c$. Now E and F being closed says that there are $\delta > 0$ and $\eta > 0$ such that the balls $B_\delta(x) \subset E^c$ and $B_\eta(x) \subset F^c$. Let $\epsilon = \min\{\delta, \eta\}$. Then $B_\epsilon(x) \subset B_\delta(x) \subset E^c$ and $B_\epsilon(x) \subset B_\eta(x) \subset F^c$. It follows that $B_\epsilon(x) \subset E^c \cap F^c = (E \cup F)^c$. Thus $(E \cup F)^c$ is open and so $E \cup F$ is closed.

2. *Let* $\{u_n\}$ *be a sequence and u be a point in* $\mathbf{R}^d$. *Define:* $u_n \to u$ *as* $n \to \infty$. *Assume* $u_n \to u$ *and* $v_n \to v$ *as* $n \to \infty$ *in* $\mathbf{R}^d$ *and let* $a \in \mathbf{R}^d$. *Prove using just the definition and vector space properties (but not the Main Limit Theorem) that*

   $$(u_n - a) \cdot v_n \to (u - a) \cdot v \qquad \text{as } n \to \infty.$$

   A sequence converges $u_n \to u$ as $n \to \infty$ in $\mathbf{R}^d$ if for every $\epsilon > 0$ there is $N \in \mathbf{R}$ such that

   $$\|u_n - u\| < \epsilon \qquad \text{whenever } n > N.$$

   First we find a bound for $\|u_n - a\|$. Since $u_n \to u$ as $n \to \infty$, for $\epsilon = 1$ there is $N_1 \in \mathbf{R}$ so that

   $$\|u_n - u\| < 1 \qquad \text{whenever } n > N_1.$$

   It follows that

   $$\|u_n - a\| = \|u_n - u + u - a\| \le \|u_n - u\| + \|u - a\| < 1 + \|u - a\| \qquad \text{whenever } n > N_1.$$

   Then we prove the limit is as claimed. Choose $\epsilon > 0$. Since $u_n \to u$ as $n \to \infty$, there is $N_2 \in \mathbf{R}$ such that

   $$\|u_n - u\| < \frac{\epsilon}{2(1 + \|v\|)} \qquad \text{whenever } n > N_2.$$

   Since $v_n \to v$ as $n \to \infty$, there is $N_3 \in \mathbf{R}$ such that

   $$\|v_n - v\| < \frac{\epsilon}{2(1 + \|u - a\|)} \qquad \text{whenever } n > N_3.$$

   Let $N = \max\{N_1, N_2, N_3\}$. If $n > N$ we have using the triangle and the Schwarz inequalities,

   $$\begin{aligned} |(u_n - a) \cdot v_n - (u - a) \cdot v| &= |(u_n - a) \cdot v_n - (u_n - a) \cdot v + (u_n - a) \cdot v - (u - a) \cdot v| \\ &= |(u_n - a) \cdot (v_n - v) + (u_n - a - u + a) \cdot v| \\ &\le |(u_n - a) \cdot (v_n - v)| + |(u_n - a - u + a) \cdot v| \\ &\le \|u_n - a\| \, \|v_n - v\| + \|u_n - u\| \, \|v\| \\ &\le (1 + \|u - a\|) \frac{\epsilon}{2(1 + \|u - a\|)} + \|v\| \frac{\epsilon}{2(1 + \|v\|)} < \epsilon, \end{aligned}$$

   proving $(u_n - a) \cdot v_n \to (u - a) \cdot v$ as $n \to \infty$ as claimed.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *Let* $\|(u_1, u_2)\| = |u_1 - u_2|$. *Then the function* $\|(u_1, u_2)\|$ *is a norm on* $\mathbf{R}^2$.
   FALSE. This candidate is not positive definite. Indeed, $\|(1,1)\| = |1-1| = 0$, however, $(1,1) \neq (0,0)$.

   (b) *Let* $E \subset \mathbf{R}^d$. *Then the interior of the boundary is empty* $(\partial E)^\circ = \emptyset$.
   FALSE. In $\mathbf{R}$, consider the set of rationals in the unit interval $E = \mathbb{Q} \cap [0,1]$. Then $\partial E = [0,1]$ and $(\partial E)^\circ = (0,1)$.

   (c) $f(x) \sim \displaystyle\sum_{k=0}^{\infty} \frac{\cos(kx)}{2^k}$ *is differentiable for all* $x \in \mathbf{R}$.
   TRUE. We apply the Theorem from problem 152[3]. The summands are differentiable functions

$$f_k(x) = \frac{\cos kx}{2^k}, \qquad f_k'(x) = -\frac{k \sin kx}{2^k}.$$

   At $x = 0$, $f_k(0) = 2^{-k}$ which is a summable geometric series. The derivatives satisfy

$$|f_k'(x)| \le \frac{k}{2^k} = M_k \qquad \text{for all } x \text{ and all } k.$$

   But $\sum_{k=1}^{\infty} M_k < \infty$ is summable. It follows that $f(x)$ converges everywhere and that it is differentiable and its derivative is given by the convergent series of termwise derivatives

$$f'(x) = -\sum_{k=1}^{\infty} \frac{k \sin kx}{2^k}.$$

4. *Let* $K \subset \mathbf{R}^d$. *Define: the set* K *is* compact. *Determine whether* $E = \left\{ \left(\frac{1}{n}, \frac{1}{n^2} \right) : n \in \mathbb{N} \right\}$ *is compact in* $\mathbf{R}^2$. *Prove your answer directly from the definition (a) without using the Heine Borel Theorem.*

A set $K \subset \mathbf{R}^d$ is *compact* if every cover has a finite subcover. That is, if $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ is a collection of open sets such that $K \subset \cup_{\alpha \in A} \mathcal{O}_\alpha$ then there are finitely many indices $\alpha_1, \ldots, \alpha_p$ such that $K \subset \mathcal{O}_{\alpha_1} \cup \cdots \cup \mathcal{O}_{\alpha_p}$.

The given set E is NOT COMPACT. We produce a cover that does not have a finite subcover. We find a collection of pairwise disjoint open sets each of which contains exactly one of the points of E. Then a finite subcover only covers finitely many of the points but not all of E. For example, if we choose points $c_0, c_1, c_2, \ldots$ such that

$$c_0 > \frac{1}{1} > c_1 > \frac{1}{2} > \cdots > c_{n-1} > \frac{1}{n} > c_n > \cdots$$

then we may take open sets $\mathcal{O}_n = \{(x,y) \in \mathbf{R}^2 : c_n < x < c_{n-1} \text{ and } y \in \mathbf{R}.\}$. The point $\left(\frac{1}{n}, \frac{1}{n^2} \right)$ is in $\mathcal{O}_n$ but is not in $\mathcal{O}_i$ if $i \neq n$. One possible choice is

$$c_n = \frac{1}{n + \frac{1}{2}}.$$

5. (a) *Suppose that* $|a_k| \le |b_k|$ *for* k *large. Prove that if* $\sum_{k=0}^{\infty} b_k\, x^k$ *converges in an open interval* I *then* $\sum_{k=0}^{\infty} a_k\, x^k$ *also converges on* I.

The power series $\sum_{k=0}^{\infty} b_k\, x^k$ centered at zero converges in an interval $(-R, R)$ where

$$\frac{1}{R} = \limsup_{k\to\infty} |b_k|^{\frac{1}{k}}.$$

The power series $\sum_{k=0}^{\infty} a_k\, x^k$ centered at zero converges in an interval $(-R', R')$ where

$$\frac{1}{R'} = \limsup_{k\to\infty} |a_k|^{\frac{1}{k}} \le \limsup_{k\to\infty} |b_k|^{\frac{1}{k}} = \frac{1}{R},$$

where we have used $|a_k| \le |b_k|$ for k large. It follows that $R' \ge R$ so that $\sum_{k=0}^{\infty} a_k\, x^k$ converges at least in the same interval $(-R', R') \supset (-R, R)$.

(b) *Suppose that* g *is infinitely differentiable function on* $(a-r, a+r)$ *and there is a constant* M *such that the* k*th derivative*

$$|g^{(k)}(z)| \le \frac{Mk!}{r^k} \qquad \text{for all } k = 0, 1, 2, 3, \ldots \text{ and all } z \in (a-r, a+r).$$

Show that the Taylor Series for $g(x)$ *at* a *converges to* g *on* $(a-r, a+r)$.

Let

$$g_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

be the nth Taylor polynomial. The error is given by

$$g(x) - g_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is a point between a and $x \in (a-r, a+r)$. Estimating the error using the given inequality,

$$\begin{aligned}
|g(x) - g_n(x)| &\le \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right| \\
&\le \frac{M(n+1)!}{r^{n+1}(n+1)!}|x-a|^{n+1} \\
&= \frac{M|x-a|^{n+1}}{r^{n+1}} \to 0
\end{aligned}$$

as $n \to \infty$ since $|x-a| < r$. It follows that for $x \in (a-r, a+r)$, the Taylor polymomial converges to the function: $g_n(x) \to g(x)$ as $n \to \infty$.