Solutions of Homework N Problems Math 3220 §2, Spring 2018

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Here are the solutions to some homework problems from Joe Taylor's text, *Foundations of Analysis*. Students often had the right idea to solve the problem, but did not write complete and careful solutions.

1. *Let $B \subset \mathbf{R}^d$ be a compact Jordan region and $f, g : B \to \mathbf{R}$ be continuous functions such that $g(x) \le f(x)$ for all $x \in B$. Define the region*

$$A = \{(x,t) \in \mathbf{R}^{d+1} : \ x \in B \text{ and } g(x) \le t \le f(x)\}.$$

Show that A is a Jordan region. [Taylor, p. 288, Problem 12.]

Let R be a big aligned rectangle such that $B \subset R$. Since f, g are continuous on a compact set B, they are bounded: there is $M < \infty$ such that $-M < g(x) \le f(x) \le M$ for all $x \in B$. Thus A is a boundend set since it is a subset of the rectangle $A \times [-M, M]$.

We use the theorem that A is Jordan if and only if ∂A has volume zero. A set $E \subset \mathbf{R}^d$ has volume zero if the upper volume is zero. This is equivalent to two statements. The first is that for every $\epsilon > 0$, there is a partition $mathcal P$ of R such that $U(\chi_E, \mathcal{P}) - L(\chi_E, \mathcal{P}) < \epsilon$. The other is that for every $\epsilon > 0$ there are finitely many aligned boxes R_i such that

$$E \subset \bigcup_i R_i \qquad \text{and} \qquad \sum_i V(R_i) < \epsilon.$$

To handle ∂A we notice that

$$\partial A \subset \partial B \times [-M, M] \cup S \cup T$$

where

$$S = \{(x, g(x)) \in \mathbf{R}^{d+1} : x \in B\}, \qquad T = \{(x, f(x)) \in \mathbf{R}^{d+1} : x \in B\}.$$

Choose $\epsilon > 0$. Since B is Jordan, ∂B has volume zero. Since f, g are continuous on a compact set B, they are uniformly continuous, so there is $\delta > 0$ such that

$$|f(x) - f(y)| + |g(x) - g(y)| < \epsilon \qquad \text{whenever } x, y \in B \text{ and } |x - y| < \delta.$$

Now choose a partition of R such that the mesh size (diameter of the maximal subrectangle) is less than δ and

$$U(\chi_{\partial B}, \mathcal{P}) - L(\chi_{\partial B}, \mathcal{P}) < \epsilon.$$

Now let us construct a cover of ∂A by rectangles whose total colume is negligible. We ignore rectangles of $\mathcal{P}$ such that $R_i \cap B = \emptyset$.. For rectangles such that $R_i \cap \partial B \ne \emptyset$, we consider

$R_i \times [-M, M]$ which will cover $\partial B \times [-M, M]$. For interior rectangles $R_i \subset B^\circ$, we define

$$m_i = \inf_{R_i} g, \qquad M_i = \sup_{R_i} g$$
$$n_i = \inf_{R_i} f, \qquad N_i = \sup_{R_i} f$$

and consider $R_i \times [m_i, M_i]$ which will cover S, and $R_i \times [n_i, N_i]$ which will cover T. Now we have

$$\partial B \times [-M.M] \cup S \cup T \subset \bigcup_{R_i \cap \partial B \neq \emptyset} R_i \times [-M, M] \cup \bigcup_{R_i \subset B^\circ} R_i \times [m_i, M_i] \cup \bigcup_{R_i \subset B^\circ} R_i \times [n_i, N_i].$$

Using the fact that f, g are uniformly continuous and that the diameter of R_i is less than δ, since f and g take their maxima and minima in $R_i \subset B^\circ$ we have $M_i - m_i < \epsilon$ and $N_i - n_i < \epsilon$. It follows that the total volume of the covering boxes is less than

$$\begin{aligned}
\sum_{R_i \cap \partial B \neq \emptyset} & V(R_i \times [-M, M]) + \sum_{R_i \subset B^\circ} V(R_i \times [m_i, M_i]) + \sum_{R_i \subset B^\circ} V(R_i \times [n_i, N_i]) \\
&= 2M \sum_{R_i \cap \partial B \neq \emptyset} V(R_i) + \sum_{R_i \subset B^\circ} V(R_i)(M_i - m_i) + \sum_{R_i \subset B^\circ} V(R_i)(N_i - n_i) \\
&\le 2\epsilon M + \epsilon \sum_{R_i \subset B^\circ} V(R_i) + \epsilon \sum_{R_i \subset B^\circ} V(R_i) \\
&\le 2\epsilon M + \epsilon V(R) + \epsilon V(R) = (2M + 2V(R))\epsilon.
\end{aligned}$$

which is arbitrarily small for small ϵ. Thus

$$V(\partial B \times [-M.M] \cup S \cup T) = 0.$$

which implies $V(\partial A) = 0$ and A is Jordan.

2. *Suppose that $A, B \subset \mathbf{R}^d$ are Jordan regions such that $V(A \cap B) = 0$. Let $f : A \cup B \to \mathbf{R}$ be a bounded function. Show that f is integrable on A and g is integrable on B if and only if f is integrable on $A \cup B$. If this is true, then*

$$\int_A f\, dV + \int_B f\, dV = \int_{A \cup B} f\, dV.$$

[Taylor, p. 293, Problem 5.]

Let R be a big rectangle such that $A \cup B \subset R$. Recall that if $E \subset R$ then $f_E = \chi_E f$, the function that is f on E and zero off E. Let $E \subset R$ be a Jordan region. We use the theorem that necessary and sufficient for f being integrable on E is that for every $\epsilon > 0$ there is a partition $\mathcal{P}$ such that

$$U(f_E, \mathcal{P}) - L(f_E, \mathcal{P}) < \epsilon.$$

Let us assume that f is integrable on A and f is integrable on B to show f is integrable on $A \cup B$. Hence f is bounded: there is $M < \infty$ such that $|f| < M$. Moreover, we have $V(A \cap B) = 0$. Thus there are partitions $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ such that

$$\begin{aligned}
U(f_A, \mathcal{P}_1) - L(f_A, \mathcal{P}_1) &< \epsilon \\
U(f_B, \mathcal{P}_2) - L(f_B, \mathcal{P}_2) &< \epsilon \\
U(\chi_{A \cap B}, \mathcal{P}_3) - L(\chi_{A \cap B}, \mathcal{P}_3) &< \epsilon
\end{aligned}$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ and $\{R_i\} = \mathcal{P}_j$ and $\{R_i'\} = \mathcal{P}$ their rectangles. Let

$$m_i' = \inf_{R_i'} f, \qquad M_i' = \sup_{R_i'} f.$$

We compute

$$\begin{aligned}
U(f_{A\cup B}, \mathcal{P}) - L(f_{A\cup B}, \mathcal{P}) &= \sum_{R_i'} (M_i'(f_{A\cup B}) - m_i'(f_{A\cup B}))V(R_i') \\
&= \sum_{R_i' \cap B = \emptyset} (M_i'(f_{A\cup B}) - m_i'(f_{A\cup B}))V(R_i') \\
&\quad + \sum_{R_i' \cap A \cap B \neq \emptyset} (M_i'(f_{A\cup B}) - m_i'(f_{A\cup B}))V(R_i') \\
&\quad + \sum_{R_i' \cap A = \emptyset} (M_i'(f_{A\cup B}) - m_i'(f_{A\cup B}))V(R_i') \\
&\leq \sum_{R_i' \cap B = \emptyset} (M_i'(f_A) - m_i'(f_A))V(R_i') \\
&\quad + 2M \sum_{R_i' \cap A \cap B \neq \emptyset} V(R_i') \\
&\quad + \sum_{R_i' \cap A = \emptyset} (M_i'(f_B) - m_i'(f_B))V(R_i') \\
&\leq \sum_{R_i'} (M_i'(f_A) - m_i'(f_A))V(R_i') \\
&\quad + 2M \sum_{R_i' \cap A \cap B \neq \emptyset} V(R_i') \\
&\quad + \sum_{R_i'} (M_i'(f_B) - m_i'(f_B))V(R_i') \\
&\leq U(f_A, \mathcal{P}) - L(f_A, \mathcal{P}) \\
&\quad + 2M(U(\chi_{A\cap B}, \mathcal{P}) - L(\chi_{A\cap B}, \mathcal{P}) \\
&\quad + U(f_B, \mathcal{P}) - L(f_B, \mathcal{P}) \\
&\leq U(f_A, \mathcal{P}_1) - L(f_A, \mathcal{P}_1) \\
&\quad + 2M(U(\chi_{A\cap B}, \mathcal{P}_2) - L(\chi_{A\cap B}, \mathcal{P}_2) \\
&\quad + U(f_B, \mathcal{P}_3) - L(f_B, \mathcal{P}_3) \\
&\leq \epsilon + 2M\epsilon + \epsilon = (2 + 2M)\epsilon
\end{aligned}$$

where, for example, we have used

$$M_i'(f_{A\cup B}) - m_i'(f_{A\cup B}) = M_i'(f_A) - m_i'(f_A) \tag{1}$$

for rectangles $R_i' \cap B = \emptyset$ and the refinement inequality

$$U(f_B, \mathcal{P}) - L(f_B, \mathcal{P}) \leq U(f_B, \mathcal{P}_j) - L(f_B, \mathcal{P}_j). \tag{2}$$

Since ϵ was arbitrary, f is integrable on $A \cup B$.

Let us assume that f is integrable on $A \cup B$ to show f is integrable on A and f is integrable on B. We do integrable on A. Integrable on B is similar. Thus there are partitions $\mathcal{P}_1$, $\mathcal{P}_2$ such that

$$\begin{aligned}
U(f_{A\cup B}, \mathcal{P}_1) - L(f_{A\cup B}, \mathcal{P}_1) &< \epsilon \\
U(\chi_{A\cap B}, \mathcal{P}_2) - L(\chi_{A\cap B}, \mathcal{P}_2) &< \epsilon
\end{aligned}$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. We estimate

$$\begin{aligned}
U(f_A,\mathcal{P}) - L(f_A,\mathcal{P}) &= \sum_{R_i'} (M_i'(f_A) - m_i'(f_A))V(R_i') \\
&= \sum_{R_i'\cap B=\emptyset} (M_i'(f_A) - m_i'(f_A))V(R_i') \\
&\quad + \sum_{R_i'\cap A\cap B\neq\emptyset} (M_i'(f_A) - m_i'(f_A))V(R_i') \\
&\le \sum_{R_i'\cap B=\emptyset} (M_i'(f_{A\cup B}) - m_i'(f_{A\cup B}))V(R_i') \\
&\quad + 2M \sum_{R_i'\cap A\cap B\neq\emptyset} V(R_i') \\
&\le \sum_{R_i'} (M_i'(f_{A\cup B}) - m_i'(f_{A\cup B}))V(R_i') \\
&\quad + 2M \sum_{R_i'\cap A\cap B\neq\emptyset} V(R_i') \\
&= U(f_{A\cup B},\mathcal{P}) - L(f_{A\cup B},\mathcal{P}) \\
&\quad + U(\chi_{A\cap B},\mathcal{P}) - L(\chi_{A\cap B},\mathcal{P}) \\
&\le U(f_{A\cup B},\mathcal{P}_1) - L(f_{A\cup B},\mathcal{P}_1) \\
&\quad + U(\chi_{A\cap B},\mathcal{P}_2) - L(\chi_{A\cap B},\mathcal{P}_2) \\
&\le \epsilon + 2M\epsilon = (1+2M)\epsilon.
\end{aligned}$$

where we have used (1) for $R_i'\cap B=\emptyset$ and (2). Since ϵ was arbitrary, f is integrable on A.

Finally, if either f is integrable on $A\cup B$ or f is integrable on both A and B then it follows that f is integrable on all sets A, B, $A\cup B$ and $A\cap B$ so that the functions add

$$f_{A\cup B} = f_A + f_B - f_{A\cap B}$$

so

$$\int_{A\cup B} f = \int_R f_{A\cup B} = \int_R f_A + \int_R f_B - \int_R f_{A\cap B} = \int_A f + \int_B -0,$$

proving the claimed equation.

3. *Let $L : \mathcal{R}^d \to \mathcal{R}^d$ be a singular linear map and $E \subset \mathbf{R}^d$ be a Jordan region. Then the volume $L(E)$ has volume zero. [Taylor, p. 314, Problem 2.]*

   The map is singular if the matrix L^T has a nontrivial null vector $w \in \mathbf{R}^d$. In other words $w^T L = 0$ or $w \bullet L(x) = 0$ for all $x \in \mathbf{R}^d$. It follows that $L(E)$ is contained in the subspace $w^\perp$. One of the components of w is nonzero (lest $w=0$), so we finish the argument in case that $w_d \neq 0$. If another component is nonzero we argue similarly. Since E is Jordan, it is bounded, and L is Lipschitz so $L(E)$ is bounded. Let $R \subset \mathbf{R}^d$ be a big rectangle of the form $R = S \times [-M, M]$ where S is a rectangle of $\mathbf{R}^{d-1}$ that contains $L(E)$. We shall show that the set $T = R \cap w^\perp$ has volume zero, and so its subset $L(E)$ has volume zero too. A point $x = (s,t)$ is in $w^\perp$ if $w \bullet x = 0$. Solving for t, we find

$$T = \left\{ (s,t) : s \in S, t = f(s) = -\frac{w_1}{w_d}s_1 - \cdots - \frac{w_{d-1}}{w_d}s_1 \right\}$$

To show $V(T) = 0$ we use the theorem that $V(T) = 0$ if and only if every $\epsilon > 0$ there are finitely many boxes R_i such that

$$T \subset \bigcup_i R_i \qquad \text{and} \qquad \sum_i V(R_i) < \epsilon.$$

Choose $\epsilon > 0$. Put

$$\lambda = |\nabla f| = \sqrt{\frac{w_1^2}{w_d^2} + \cdots + \frac{w_{d-1}^2}{w_d^2}},$$

the slope of T. Choose a partition $\mathcal{P} = \{S_i\}$ of S with mesh (maximal box diameter) less than $\dfrac{\epsilon}{V(S)\lambda + 1}$. Let

$$m_i = \inf_{S_i} f, \qquad m_i = \sup_{S_i} f$$

Then T is covered by the rectangles $S_i \times [m_i, M_i]$. But $M_i - m_i < \dfrac{\epsilon}{V(S)}$ since points of S_i are less than $\dfrac{\epsilon}{V(S)\lambda + 1}$ apart. Hence

$$T \subset \bigcup_i S_i \times [m_i, M_i] \qquad \text{and} \qquad \sum_i V(S_i \times [m_i, M_i]) < \frac{\epsilon}{V(S)} \sum_i V(S_i) = \epsilon.$$

Since ϵ was arbitrary, T has volume zero and so does its subset $L(E)$.