Math 3220 § 2. **Third Midterm Exam** Name: Solutions
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1. *Let $f : \mathbf{R}^p \to \mathbf{R}^q$ be a function. State the definition: f is a differentiable at $a \in \mathbf{R}^p$. Using just the definition, determine whether $f : \mathbf{R}^2 \to \mathbf{R}^2$ is differentiable at $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$ and prove your answer, where*
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-y \\ x^2 \end{pmatrix}.$$

$F : \mathbf{R}^p \to \mathbf{R}^q$ is differentiable at $a \in \mathbf{R}^p$ if there is a $q \times p$ matrix M such that
$$\lim_{h\to 0} \frac{F(a+h) - F(a) - Mh}{|h|} = 0.$$

We show that f is differentiable at $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$. If the given function were differentiable, then the matrix is given by the Jacobian
$$M = \begin{pmatrix} \dfrac{\partial f_1}{\partial x}, \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x}, \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2x & 0 \end{pmatrix}, \qquad M\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} -k \\ 2xh \end{pmatrix}$$

Then the norm of the difference quotient limits to zero. Indeed,
$$\begin{aligned} \frac{|f(x+h, y+k) - f(x,y) - M\binom{h}{k}|}{\sqrt{h^2+k^2}} &= \frac{1}{\sqrt{h^2+k^2}} \left| \begin{pmatrix} 1-y-k-(1-y)-(-k) \\ x^2+2xh+h^2-x^2-2xh \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{h^2+k^2}} \left| \begin{pmatrix} 0 \\ h^2 \end{pmatrix} \right| \le \frac{h^2+k^2}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2} \to 0 \end{aligned}$$

as $(h,k) \to (0,0)$. Thus f is differentiable at $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$ and $df\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2x & 0 \end{pmatrix}$.

2. *Let* $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $f(\mathbf{x}) = \begin{pmatrix} yz \\ x \\ y \end{pmatrix}$, $g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v^2 \\ u+v \\ u^2 \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = g(\mathbf{a}) = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$.

Find $df(\mathbf{x})$, $dg(\mathbf{a})$, *and* $d(f \circ g)(\mathbf{a})$. *Determine all* $\mathbf{x}$ *where* f *has a local inverse in the neighborhood of* $\mathbf{x}$. *Does it have a local inverse in a neighborhood of the vector* $\mathbf{b}$? *If so, what is* $d(f^{-1})(f(\mathbf{b}))$? *Be sure to verify that the hypotheses are fulfilled for any theorems you use.*

By the chain rule,

$$df(\mathbf{x}) = \begin{pmatrix} 0 & z & y \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad df(\mathbf{b}) = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$dg\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 2v \\ 1 & 1 \\ 2u & 0 \end{pmatrix}, \qquad dg(\mathbf{a}) = \begin{pmatrix} 0 & 4 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$$

$$d(f \circ g)(\mathbf{a}) = df(g(\mathbf{a}))\, dg(\mathbf{a}) = df(\mathbf{b})\, dg(\mathbf{a}) = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 0 & 4 \\ 1 & 1 \end{pmatrix}$$

The function $f(\mathbf{x})$ is polynomial, so is $\mathcal{C}^1$ which is the first condition for the Inverse Function Theorem. The determinant of the differential $\det(df(\mathbf{x})) = y$ so that the inverse function theorem applies when $df(\mathbf{x})$ is invertible, which is when $y \neq 0$. At $\mathbf{a}$ this holds. Thus f is locally invertible at $\mathbf{a}$: there are open sets $\mathbf{a} \in U$ and $\mathbf{b} \in V$ and a $\mathcal{C}^1$ function $f^{-1} = g : V \to U$ such that $g(f(x)) = x$ all $x \in U$ and $f(g(y)) = y$ all $y \in V$. Moreover, $dg(y) = df(g(y))^{-1}$ for all $y \in V$. In particular, using, *e.g.,* Cramer's rule

$$d(f^{-1})(f(\mathbf{b})) = (df(g(f(\mathbf{b})))^{-1} = (df(\mathbf{b}))^{-1} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix}$$

as one can eaasily check.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *Suppose $f(x,y)$ is a continuous function such that all first partial derivatives exist at all $(x,y)\in\mathbf{R}^2$. Then f is differentiable at all $(x,y)\in\mathbf{R}^2$.*

   FALSE. The function

$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & \text{if } (x,y)\neq(0,0);\\[2ex] 0, & \text{if } (x,y)=(0,0).\end{cases}$$

   is continuous and has partial derivatives everywhere, but is not differentiable at $(0,0)$. For $(x,y)\neq(0,0)$, the function is a quotient of differentiable functions, so continuous and first partials exist and are continuous. Note that $|f(x,y)-f(0,0)|=|f(x,y)|\le\sqrt{x^2+y^2}$ so f is continuous at $(0,0)$. The partial derivatives exist at $(0,0)$. In particular

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to0}\frac{f(h+0,0)-f(0,0)}{h}=\lim_{h\to0}\frac{h^3}{h^3}=1,$$
$$\frac{\partial f}{\partial y}(0,0)=\lim_{k\to0}\frac{f(0,k)-f(0,0)}{k}=\lim_{h\to0}\frac{0}{k^3}=0.$$

   If f were differentiable at $(0,0)$, its differential would be the Jacobian matrix $df(0,0)=(1,0)$. However, the limit of the difference quotient

$$\begin{aligned}&\lim_{(h,k)\to(0,0)}\frac{f(h+0,k+0)-f(0,0)-df(0,0)\binom{h}{k}}{|(h,k)|}\\ &=\lim_{(h,k)\to(0,0)}\frac{\dfrac{h^3}{h^2+k^2}-h}{\sqrt{h^2+k^2}}\\ &=\lim_{(h,k)\to(0,0)}\frac{-hk^3}{(h^2+k^2)^{\frac32}}\end{aligned}$$

   does not exist because it takes different values for different approaches to $(0,0)$. For example if the approach is $(h,k)=(t,t)$ as $t\to0$ then the limit is $-2^{-\frac32}$ whereas if the approach is $(t,0)$ then the limit is 0. Thus the function is not differentiable at $(0,0)$.

   (b) *Suppose $U\subset\mathbf{R}^2$ is open and $f\in\mathcal{C}^1(U,\mathbf{R}^2)$ satisfies $df(x)$ is invertible for all $x\in U$. Then f is one-to-one on U.*

   FALSE. Consider the map on open $D\subset\mathbf{R}^2$ to $\mathbf{R}^2$ given by

$$f\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x^2-y^2\\2xy\end{pmatrix},\qquad \begin{pmatrix}x\\y\end{pmatrix}\in D=\left\{\begin{pmatrix}x\\y\end{pmatrix}\in\mathbf{R}^2:\ \text{if } x\le0 \text{ then } y\neq0\right\}.$$

   Then f is not one-to-one on D because, for example $f\begin{pmatrix}0\\1\end{pmatrix}=f\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$. However, the differential is

$$df\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x&-2y\\2y&2x\end{pmatrix}$$

   whose determinant is $4(x^2+y^2)$ which does not vanish in D.

(c) *Suppose $f, g \in \mathcal{C}^1(\mathbf{R}^p, \mathbf{R}^p)$ such that $f \circ g(x) = x$ for all $x \in \mathbf{R}^p$. Then $dg(x)$ is invertible for all $x \in \mathbf{R}^p$.*

TRUE. Differentiating $f \circ g(x) = x$ we get from the chain rule

$$df(g(x))\, dg(x) = I$$

which implies that the $p \times p$ matrix $dg(x)$ is invertible.

4. (a) *Let $\mathcal{S} = \{(x, y, z) \in \mathbf{R}^3 : xy + z = 7 \text{ and } y^2 - 4z = 0\}$. What is the dimension of the tangent space to $\mathcal{S}$ at the point $(3, 2, 1)$? Why? Find the tangent space at $(3, 2, 1)$.*

Let $f(x, y, z) = \begin{pmatrix} xy + z \\ y^2 - 4z \end{pmatrix}$. Then $\mathcal{S} = g^{-1}\begin{pmatrix} 7 \\ 0 \end{pmatrix}$ is a level set of three space cut by two functions. The differential

$$df(x, y, z) = \begin{pmatrix} y & x & 1 \\ \\ 0 & 2y & -4 \end{pmatrix}, \qquad df(3, 2, 1) = \begin{pmatrix} 2 & 3 & 1 \\ \\ 0 & 4 & -4 \end{pmatrix}$$

which has rank 2 at $(3, 2, 1)$. Thus the dimension of the tangent space is the dimension of the kernel of $df(3, 2, 1)$ which is the number of free variables $= 3 - \text{rank} = 1$. The affine space tangent to $\mathcal{S}$ at $(3, 2, 1)$ is

$$\begin{pmatrix} 3 \\ \\ 2 \\ \\ 1 \end{pmatrix} + \ker df(3, 2, 1) = \begin{pmatrix} 3 \\ \\ 2 \\ \\ 1 \end{pmatrix} + \left\{ t \begin{pmatrix} -2 \\ \\ 1 \\ \\ 1 \end{pmatrix} : t \in \mathbf{R} \right\}$$

(b) *Let $f(x, y) = \cos(xy)$. Find the degree $n = 2$ Taylor formula with remainder for f at the point $(0, 0)$.*

The function is C^∞ so second and third partial derivatives may be done in any order and the differentials of any order exist. We compute

$$f_x(x, y) = -\sin(xy)y, \quad f_y(x, y) = -\sin(xy)x;$$
$$f_{xx}(x, y) = -\cos(xy)y^2, \quad f_{xy}(x, y) = -\cos(xy)xy - \sin(xy), \quad f_{yy}(x, y) = -\cos(xy)x^2;$$
$$f_{xxx}(x, y) = \sin(xy)y^3, \quad f_{xxy}(x, y) = \sin(xy)xy^2 - 2\cos(xy)y,$$
$$f_{xyy}(x, y) = \sin(xy)x^2y - 2\cos(xy)x, \quad f_{yyy}(x, y) = \sin(xy)x^3.$$

which are all zero if $(x, y) = (0, 0)$. Hence $f(0, 0) = 1$, and both $df(0, 0)$ and $d^2 f(0, 0)$ vanish. Letting $z = \binom{x-0}{y-0}$, it follows that Taylor's expansion to second order is

$$f(x, y) = f(0, 0) + df(0, 0)(z) + \frac{1}{2} d^2 f(0, 0)(z, z) + R_2(x, y) \quad = 1 + 0 + 0 + R_2(x, y)$$

where

$$\begin{aligned} R_2(x, y) &= \frac{1}{6}\Big\{ f_{xxx}(h, k)x^3 + 3f_{xxy}(h, k)x^2y + 3f_{xyy}(h, k)xy^2 + f_{yyy}(h, k)y^3 \Big\} \\ &= \frac{1}{6}\Big\{ \Big[\sin(hk)k^3\Big]x^3 + 3\Big[\sin(hk)hk^2 - 2\cos(hk)k\Big]x^2y \\ &\qquad + 3\Big[\sin(hk)h^2k - 2\cos(hk)h\Big]xy^2 + \Big[\sin(hk)h^3\Big]y^3 \Big\} \end{aligned}$$

and where (h, k) is some point on the line segment from $(0, 0)$ to (x, y).

5. *Find the minimum of* $f(x,y,z) = x^2 + y^2 + z^2$ *subject to* $x - y = 1$ *and* $y^2 - z^2 = 1$.

Let $g(x,y,z) = x - y - 1$ and $h(x,y,z) = y^2 - z^2 - 1$. The constraint set $\mathcal{S} = \{(x,y,z) : g(x,y,z) = h(x,y,z) = 0\}$ is the intersection of a hyperbolic cylinder with a plane, thus consists of two hyperbolic lines in $\mathbf{R}^3$. Thus we expect at least one critical point of f on each nappe of the hyperbola. Also, f is unbounded on $\mathcal{S}$, so there are no maxima. Functions here are smooth. Necessary conditions are given by Lagrange multipliers. We seek λ and μ such that

$$\begin{aligned} \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ (2x, 2y, 2z) &= \lambda(1,-1,0) + \mu(0, 2y, -2z) \end{aligned}$$

Thus we must solve the five equations for x, y, z, λ, μ

$$2x = \lambda \tag{1}$$
$$2y = -\lambda + 2\mu y \tag{2}$$
$$2z = -2\mu z \tag{3}$$
$$x - y = 1 \tag{4}$$
$$y^2 - z^2 = 1. \tag{5}$$

(3) tells us that $2z(1+\mu) = 0$ so either $\mu = -1$ or $z = 0$.

If $\mu = -1$ then (2) says $4y = -\lambda$ or $y = -\dfrac{\lambda}{4}$. (4) tells us that $2x = 2 + 2y = 2 - \dfrac{\lambda}{2}$. But (1) says $\lambda = 2x$ so $\lambda = \dfrac{4}{3}$ and thus $y = -\dfrac{1}{3}$. Finally, (5) says $\dfrac{1}{9} = y^2 = 1 + z^2 \geq 1$, which is a contradiction. The case $\mu = -1$ doesn't happen.

Thus we must have $z = 0$. Then (5) says $y = \pm 1$. If $y = 1$, then (4) says $x = 1 + y = 2$ so at this critical point $f(2,1,0) = 5$. If $y = -1$ then (4) says $x = 1 + y = 0$ so at this critical point, $f(0,-1,0) = 1$. Since we have found all critical points, the minimizer occurs at $(0,-1,0)$ where $f = 1$.