Math 3220 § 2. **Second Midterm Exam** Name: Solutions
Treibergs February 28, 2018

1. *Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a function. State the definition: f is a uniformly continuous on $\mathbf{R}^2$. Determine whether f is unformly continuous on $\mathbf{R}^2$ and prove your answer, where*

$$f(x,y) = (\ 1+y,\ \sin x\ )$$

$f : \mathbf{R}^2 \to \mathbf{R}^2$ is *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f(x_1,y_1) - f(x_2,y_2)| < \epsilon \quad \text{whenever } (x_1,y_1),(x_2,y_2) \in \mathbf{R}^2 \text{ and } |(x_1,y_1)-(x_2,y_2)| < \delta.$$

We now show that $f(x,y)$ is uniformly continuous. Choose $\epsilon > 0$. Let $\delta = \epsilon$. Then for any $(x_1,y_1),(x_2,y_2) \in \mathbf{R}^2$ such that $|(x_1,y_1)-(x_2,y_2)| < \delta$ we have

$$\begin{aligned} |f(x_1,y_1) - f(x_2,y_2)| &= |(1+y_1, \sin x_1) - (1+y_2, \sin x_2)| \\ &= |(y_1 - y_2,\ \sin x_1 - \sin x_2)| \\ &= \sqrt{|y_1-y_2|^2 + |\sin x_1 - \sin x_2|^2} \\ &\le \sqrt{|y_1-y_2|^2 + |x_1-x_2|^2} \\ &= |(x_1,y_1)-(x_2,y_2)| < \delta = \epsilon, \end{aligned}$$

where we have used $|\sin x_1 - \sin x_2| \le |x_1 - x_2|$.

2. *Let $f : \mathbf{R}^p \to \mathbf{R}^q$. State the definition: f is* continuous. *State the definition: $G \subset \mathbf{R}^q$ is* open. *Suppose $G \subset \mathbf{R}^q$ is open and f is continuous. Show that $f^{-1}(G)$ is open.*

$f : \mathbf{R}^p \to \mathbf{R}^q$ is *continuous* if for every $a \in \mathbf{R}^p$ and every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever } x \in \mathbf{R}^p \text{ and } |x - a| < \delta.$$

$G \subset \mathbf{R}^q$ is *open* if for every $y \in G$ there is an $\epsilon > 0$ so that $B_\epsilon(y) \subset G$.

To show $f^{-1}(G)$ is open, choose $x \in f^{-1}(G)$. Thus $y = f(x) \in G$. Since G is open, there is $\epsilon > 0$ so that $B_\epsilon(y) \subset G$. By the continuity, there is $\delta > 0$ so that for every $z \in \mathbf{R}^p$ such that $|z - x| < \delta$ we have $|f(z) - f(x)| < \epsilon$. In other words $f(B_\delta(x)) \subset B_\epsilon(y) \subset G$. But this says $B_\delta(x) \subset f^{-1}(G)$. But since we were able to find a ball neighborhood about every $x \in G$ that is in G, this says that $f^{-1}(G)$ is open.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *Suppose $C \subset \mathbf{R}^2$ is connected and $f : \mathbf{R}^2 \to \mathbf{R}^2$ is continuous. Then $f^{-1}(C)$ is connected.*

   False. Let $C = \{(0,0)\}$ be a single point set which is connected and $f(x,y) = (\sin x, \sin y)$. Then $f^{-1}(C) = \{(\pi k, \pi j) : j,k \in \mathbf{Z}\}$, the doubly infinite lattice in $\mathbf{R}^2$ which is not connected.

   (b) *Let $K \subset \mathbf{R}^2$ be a compact set. Then every point of K is a limit point of K.*

   False. The set $K = \overline{B_1((0,0))} \cup \{(3,0)\}$ consists of the closed ball together with the isolated point $(3,0)$. It is closed and bounded, therefore compact. But no ball with $r < 2$ around $(3,0)$ meets a point of K other than itself, so that $(3,0)$ is not a limit point of K.

   (c) *Suppose that $C \subset \mathbf{R}^p$ is closed and $f : C \to \mathbf{R}^q$ is uniformly continuous. Then $f(C)$ is closed.*

   False. The map $f(x) = \operatorname{Atn}(x)$ is uniformly continuous function (since $|f'(x)| \le 1$) from $\mathbf{R}$, which is closed, to $\mathbf{R}$ but $f(\mathbf{R}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is not closed in $\mathbf{R}$.

4. *Let $D \subset \mathbf{R}^p$ and $f, f_n : D \to \mathbf{R}^q$ be transformations. State the definition:* $\{f_n\}$ converges uniformly *to f on D. Suppose for all sequences $\{x_n\} \subset D$ we have $f_n(x_n) \to 0$ as $n \to \infty$. Show that then $f_n \to 0$ uniformly in D.*

   $f_n \to f$ *converges uniformly on D* means that for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ such that for all all $n \ge N$ and $x \in D$,
   $$|f_n(x) - f(x)| < \epsilon.$$
   To prove the statement, argue by contrapositive. The statement that f_n does not converge uniformly to 0 is: there is an $\epsilon_0 > 0$ such that for every $N \in \mathbf{R}$ there is an $n \ge N$ and an $x \in D$ such that
   $$|f_n(x) - f(x)| \ge \epsilon_0.$$
   We construct a sequence in D inductively using this. For $N = 1$ then there is $n_1 \ge 1$ and an $x_{n_1} \in D$ such that
   $$|f_{n_1}(x_{n_1}) - 0| \ge \epsilon_0.$$
   Now let $N = n_1 + 1$. There is an $n_2 \ge N = n_1 + 1$ and $x_{n_2} \in D$ such that
   $$|f_{n_2}(x_{n_2}) - 0| \ge \epsilon_0.$$
   Continuing in this way assume that $n_1 < \ldots < n_p$ and $x_{n_1}, \ldots, x_{n_p}$ have been chosen. Then for $N = n_p + 1$ there is an $n_{p+1} \ge N > n_p$ and $x_{n_{p+1}} \in D$ such that
   $$|f_{n_{p+1}}(x_{n_{p+1}}) - 0| \ge \epsilon_0.$$
   If $j \in \mathbf{N}$ is not one of the n_i's, we let x_j be any element of D. Thus we have constructed a sequence $\{x_k\}$ in D that has a subsequence $\{x_{n_p}\}$ such that $\{f_{n_p}(x_{n_p})\}$ does not converge to 0. Thus the entire sequence $\{x_n\}$ does not converge to zero. Hence the hypothesis is false, as to be shown.

5. *Let $f : \mathbf{R}^p \to \mathbf{R}$ be a real function and $a \in \mathbf{R}^p$. State the definition: f has a* partial derivative $\dfrac{\partial f}{\partial x_j}(a)$ *with respect to the jth variable at $a \in \mathbf{R}^p$. Compute $\dfrac{\partial f}{\partial x}(x, y)$ and determine whether $\dfrac{\partial f}{\partial x}(x, y)$ is continuous at $(0, 0)$, where*
   $$f(x, y) = \begin{cases} \dfrac{x^4 + y^4}{x^2 + y^2}, & \text{if } (x, y) \ne (0, 0); \\ \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$
   The partial derivative of $f : \mathbf{R}^p \to \mathbf{R}$ at $a \in \mathbf{R}^p$ with respect to x_i is defined to be the limit of the difference quotient, if it exists exists
   $$\frac{\partial f}{\partial x_i}(a) = \lim_{h \to 0} \frac{f(a_1, a_2, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_p) - f(a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_p)}{h}$$
   For the given function at $(0, 0)$,
   $$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(x + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^4 + 0}{h \cdot (h^2 + 0)} = 0.$$
   For $(x, y) \ne (0, 0)$ we freeze y and do usual differentiation
   $$\frac{\partial f}{\partial x}(x, y) = \frac{4x^3(x^2 + y^2) - (x^4 + y^4) \cdot 2x}{(x^2 + y^2)^2} = \frac{2x^5 + 4x^3y^4 - 2xy^4}{(x^2 + y^2)^2}$$

To see that this function is continuous at $(0,0)$, let $r = \sqrt{x^2+y^2}$. Because $|x| \leq r$ and $|y| \leq r$ we have

$$\left|\frac{\partial f}{\partial x}(0,0) - \frac{\partial f}{\partial x}(x,y)\right| = \left|\frac{2x^5+4x^3y^4-2xy^4}{(x^2+y^2)^2} - 0\right| \leq \frac{2r^5+4r^5+2r^5}{r^4} = 8r \to 0$$

as $(x,y) \to (0,0)$.