Math 3220 § 2. **First Midterm Exam** Name: Solutions
Treibergs January 31, 2018

1. *Let $F, G \subset \mathbf{R}^d$. Define: G is an* open set. *Define: F is a* closed set. *Using just your definitions, show that if F is closed and G is open then $F\backslash G$ is closed.*

   G is open if for every $x \in G$ there is $r > 0$ so that the ball of radius r about x satisfies $B_r(x) \subset G$. F is closed if the complement F^c is open.

   To show $F\backslash G$ is closed, it suffices to show that $(F\backslash G)^c$ is open. But, by De Morgan's law,

   $$(F\backslash G)^c = (F \cap G^c)^c = F^c \cup G$$

   where F^c is open by the assumption that F is closed and G is open by assumption. The union of two opens is open. To see it choose $x \in F^c \cup G$. Either $x \in F^c$ or $x \in G$. In the first case, there is $r > 0$ so that $B_r(x) \subset F^c \subset F^c \cup G$ since F^c is open. In the second case, there is $r > 0$ so that $B_r(x) \subset G \subset F^c \cup G$ since G is open. Thus we have shown for every $x \in (F\backslash G)^c$ there is an $r > 0$ so that $B_r(x) \subset (F\backslash G)^c$. This says $(F\backslash G)^c$ is open, thus $F\backslash G$ is closed.

2. *The cross product for $u, v \in \mathbf{R}^3$ is defined by*

   $$u \times v := (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

   It satisfies $\|u \times v\| \le \|u\|\,\|v\|$ for all $u, v \in \mathbf{R}^3$. Let $\{u_n\}$ and u be a sequence and point in $\mathbf{R}^d$. Define: $u_n \to u$ as $n \to \infty$. Assume $u_n \to u$ and $v_n \to v$ as $n \to \infty$ in $\mathbf{R}^3$. Prove using just the definition and vector space properties that that

   $$u_n \times v_n \to u \times v \qquad \text{as } n \to \infty.$$

   $u_n \to u$ as $n \to \infty$ means for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ so that

   $$\|u_n - u\| < \epsilon \qquad \text{whenever } n \ge N.$$

   First, u_n is convergent, thus bounded. Indeed, by convergence, for $\epsilon = 1$, there is an $N_1 \in \mathbf{R}$ so that

   $$\|u_n - u\| < 1 \qquad \text{whenever } n \ge N_1.$$

   Hence, if $n \ge N_1$ we have

   $$\|u_n\| = \|u_n - u + u\| \le \|u_n - u\| + \|u\| \le 1 + \|u\|.$$

   Choose $\epsilon > 0$ to show $u_n \times v_n \to u \times v$ as $n \to \infty$. Since $u_n \to u$, for every $\epsilon > 0$ there is $N_2 \in \mathbf{R}$ so that

   $$\|u_n - u\| < \frac{\epsilon}{2 + \|u\| + \|v\|} \qquad \text{whenever } n \ge N_2.$$

   Since $v_n \to v$, there is $N_3 \in \mathbf{R}$ so that

   $$\|v_n - v\| < \frac{\epsilon}{2 + \|u\| + \|v\|} \qquad \text{whenever } n \ge N_3.$$

   Let $N = \max\{N_1, N_2, N_3\}$. If $n \ge N$ then adding and subtracting cross terms, using the triangle inequality, the linearity of u and v in $u \times v$ and the upper bound we get

   $$\begin{aligned}\|u_n \times v_n - u \times v\| &= \|u_n \times v_n - u_n \times v + u_n \times v - u \times v\| \\ &\le \|u_n \times v_n - u_n \times v\| + \|u_n \times v - u \times v\| \\ &\le \|u_n \times (v_n - v)\| + \|(u_n - u) \times v\| \\ &\le \|u_n\|\,\|v_n - v\| + \|u_n - u\|\,\|v\| \\ &\le \frac{(1 + \|u\|)\epsilon}{2 + \|u\| + \|v\|} + \frac{\|v\|\epsilon}{2 + \|u\| + \|v\|} < \epsilon.\end{aligned}$$

Or you could argue that the components of the cross product are differences of products of converging scalar sequences which arise as components of converging vector sequences. Hence they converge by the main limit theorem for scalar sequences.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *Let $\|u\|$ denote the Euclidean norm on $\mathbf{R}^d$. Then the function $\delta(x, y) = \|x - y\|^2$ is a metric on $\mathbf{R}^d$.*

   FALSE. The triangle inequality fails. Take $u \in \mathbf{R}^d$ such that $\|u\| = 1$. Let $v = \frac{1}{2}u$. Then

$$\delta(0, v) + \delta(v, u) = \|v - 0\|^2 + \|u - v\|^2 = \frac{1}{2^2}\|u\|^2 + \frac{1}{2^2}\|u\|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

   is less than $\delta(0, u) = \|u - 0\|^2 = 1$.

   (b) *Suppose $E \subset \mathbf{R}^2$ is not open. Then $E = \overline{E}$.*

   FALSE. The set $E = [0, 1) \times [0, 1)$ is not open since no ball about $(0, 0)$ is in E. But $\overline{E} = [0, 1] \times [0, 1]$ does not equal E.

   (c) *Let $\{x_n\} \subset \partial E$ be a subsequence contained in the boundary of the set $E \subset \mathbf{R}^d$. If it converges in $\mathbf{R}^d$, then $\lim_{n\to\infty} x_n \in \partial E$.*

   TRUE. It is a consequence of the closedness of ∂E. To see the closedness, $\partial E = \overline{E} \backslash E^\circ = \overline{E} \cap (E^\circ)^c$ is the intersection of closed sets, so is closed. Now $\{x_n\} \subset \partial E$ so $\lim_{n\to\infty} x_n \in \partial E$ since closed sets contain their limit points.

4. *Let $K \subset \mathbf{R}^d$. Define: the set K is compact. Let $K \subset \mathbf{R}^d$ be a compact set and $C \subset K$ be a closed set. Show directly from the definition (without using the Heine-Borel Theorem) that C is a compact set.*

   We wish to show that any open cover of C has a finite subcover. Let $\{G_\alpha\}_{\alpha\in A}$ be an open cover of C (so $C \subset \cup_{\alpha\in A} G_\alpha$.). It may not cover K. Throw in another open set C^c to make a larger cover $\{C^c\} \cup \{G_\alpha\}_{\alpha\in A}$. Now K is covered by the enlarged cover

$$K \subset C^c \cup \left(\bigcup_{\alpha\in A} G_\alpha \right)$$

   because $K \cap C$ is covered by the original cover and if $x \in K\backslash C$ then $x \in C^c$. By the compactenss of K, there is a finite collection $\alpha_1, \alpha_2, \ldots, \alpha_p$ in A such that

$$K \subset C^c \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_p}$$

   where we've added one additional set C^c whether or not it was one of the finite list covering K. Since $C^c \cap C = \emptyset$, we may discard C^c to find

$$C \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_p}.$$

   In other words, we found a finite subcover for C, hence C is compact.

5. *Determine the radius of convergence.*

$$f(x) \sim 3 + x + 2^2x^2 + 3^2x^3 + x^4 + 2^5x^5 + 3^6x^6 + x^7 + 2^8x^8 + 3^9x^9 \cdots$$

   Suppose that $g \in \mathcal{C}^\infty(\mathbf{R})$. Assume that for some $M < \infty$, the kth derivative $|g^{(k)}(z)| \le M$ for all $k = 0, 1, 2, 3, \ldots$ and all $z \in \mathbf{R}$. Show that the Maclaurin Series for $g(x)$ converges for all $\mathbf{R}$.

The coefficients of the power series $f(x)$ are $c_0 = 3$ and for $n \geq 1$,

$$c_n = \begin{cases} 1^n, & \text{if } n \equiv 1 \mod 3; \\ 2^n, & \text{if } n \equiv 2 \mod 3; \\ 3^n, & \text{if } n \equiv 0 \mod 3. \end{cases}$$

Hence, for $n \geq 1$ we have

$$|c_n|^{\frac{1}{n}} = \begin{cases} 1, & \text{if } n \equiv 1 \mod 3; \\ 2, & \text{if } n \equiv 2 \mod 3; \\ 3, & \text{if } n \equiv 0 \mod 3. \end{cases}$$

It follows that from the formula for the radius of convergence

$$\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}} = 3$$

so $R = \dfrac{1}{3}$.

To prove that the Maclaurin series converges for all x we must show that the partial sums (the Taylor polynomials) converge to $g(x)$ as $n \to \infty$. The Taylor polynomial is given by

$$g_n(x) = g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + \cdots + \frac{1}{n!}g^{(n)}(0)x^n$$

which makes sense for all n because we assumed that g was infinitely differentiable. For fixed $x \in \mathbf{R}$, the error from Taylor's remainder formula is estimated by the given assumption

$$|g(x) - g_n(x)| = |R_n(x)| = \left| \frac{g^{(n+1)}(c_n)}{(n+1)!} x^{n+1} \right| \leq \frac{M|x|^{n+1}}{(n+1)!} \longrightarrow 0 \qquad \text{as } n \to \infty,$$

where c_n are some numbers between 0 and x. The error tends to zero since the factorial overwhelms the power. Since x was arbitrary, the Maclaurin series converges for all $x \in \mathbf{R}$.