Math 3220 § 2. **Final Exam** Name: Solutions
Treibergs $\sigma\tau$ May 1, 2018

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1. *Determine whether the following function $f : \mathbf{R}^2 \to \mathbf{R}$ is differentiable at $(0,0)$.*

$$f(x,y) = \begin{cases} \dfrac{y \sin(x^2+y^2)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

f IS DIFFERENTABLE AT $(0,0)$. Note that $f(x,0) = 0$ for all x so that the partial derivative $f_x(0,0) = 0$ and that

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)}{h} = \lim_{h\to 0} \frac{h\sin(h^2)}{h^3} = \lim_{h\to 0} \frac{\sin(h^2)}{h^2} = 1.$$

Hence if differentiable, the differential would be $df(0,0)(h,k) = k$. Plugging into the two dimensional difference quotient we see

$$\begin{aligned}
\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - df(0,0)(h,k)|}{\|(k,h)\|} &= \lim_{(h,k)\to(0,0)} \frac{\left|\dfrac{k\sin(h^2+k^2)}{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{|k\sin(h^2+k^2) - k(h^2+k^2)|}{(h^2+k^2)^{3/2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{|k|\,\left|\sin(h^2+k^2) - (h^2+k^2)\right|}{(h^2+k^2)^{3/2}} \\
&\leq \lim_{(h,k)\to(0,0)} \frac{\left|\sin(h^2+k^2) - (h^2+k^2)\right|}{h^2+k^2} = 0,
\end{aligned}$$

where we have used l'Hôpital's rule

$$\lim_{t\to 0} \frac{\sin t - t}{t} = \lim_{t\to 0} \frac{\cos t - 1}{1} = 0.$$

Hence $f(x,y)$ is differentiable at $(0,0)$ and $df(0,0)(h,k) = k$.

2. *Let $A \subseteq \mathbf{R}^d$. Define: A is a* compact set. *Let $E = \{(x,y) \in \mathbf{R}^2 : x^2 + 2y^2 = 1\}$. Is E compact? Why? For the set E, is there a point $x \in E$ closest to $(0,0)$? Why?*

A set $A \subset \mathbf{R}^d$ is *compact* if every open cover has a finite subcover. That is, if there is a family of open sets $\{U_\beta\}_{\beta\in B}$ in $\mathbf{R}^d$ such that $A \subset \cup_{\beta\in B} U_\beta$ then there are finitely many $\beta_1, \ldots, \beta_k$ such that $A \subset U_{\beta_1} \cup \cdots \cup U_{\beta_k}$.

E is compact because it is closed and bounded by the Heine-Borel Theorem. To see it's bounded, every $(x,y) \in E$ satisfies $\|(x,y)\| = \sqrt{x^2+y^2} \leq \sqrt{x^2+2y^2} = 1$. To see it's closed, consider a sequence $(x_n,y_n) \in E$ which converges $(x_n,y_n) \to (x,y)$ as $n \to \infty$ in $\mathbf{R}^2$, we have by continuity $1 = \lim_{n\to\infty} x_n^2 + 2y_n^2 = x^2 + 2y^2$ so $(x,y) \in E$. But since E contains its limit points, it is closed.

The distance from (x,y) to $(0,0)$ is given by $f(x,y) = \|(x,y)\|$ which is continuous because, by the reverse triangle inequality, $\big|\|(x,y)\| - \|(p,q)\|\big| \leq \|(x-p, y-q)\|$ so $f(x,y)$ is Lipschitz, hence continuous. A continuous function takes its minimum on a compact set. Thus there is an $(x_0,y_0) \in E$ such that $f(x_0,y_0) = \inf\{f(x,y) : (x,y) \in E\}$. Thus $(x_0,y_0) \in E$ is a closest point of E to the origin.

3. *Let*

$$f(x,y,z,u,v) = x + 2y + 3u - zv,$$
$$g(x,y,z,u,v) = xy + zu + v,$$
$$S = \{(x,y,z,u,v) \in \mathbf{R}^5 : f(x,y,z,u,v) = 2 \text{ and } g(x,y,z,u,v) = 19\}.$$

Define: S is a regular parameterized p-dimensional surface. Show that S is a regular parameterized p-dimensional surface. Is $S \neq \emptyset$? What is p?

$S \subset \mathbf{R}^n$ is a *regular parameterized p dimensional surface* if for every $P_0 \in S$ there is an open subset $U \subset \mathbf{R}^n$ such that $P_0 \in U$, an open $V \subset \mathbf{R}^p$ and a one-to-one $\mathcal{C}^1$ function $G : V \to \mathbf{R}^n$ where $P_0 = G(Q_0)$ for some $Q_0 \in V$ and image $G(V) = U \cap S$ such that $dG(Q)$ has rank p at all $Q \in V$. The image $G(V) \subset S$ is called a local $\mathcal{C}^1$ parameterization of S.

In this case, S is a $p = 3$ dimensional parameterized surface in $\mathbf{R}^5$. To show S is not empty, we solve the equations. Setting $x = y = 0$ and $z = 1$, the equations reduce to

$$2 = f(0,0,1,u,v) = 3u - v,$$
$$19 = g(0,0,1,u,v) = u + v,$$

whose solution is $u = 5.25$ and $v = 13.75$. Thus $(0,0,1,5.25,13.75) \in S$.

Since $F(x,y,z,u,v) = (f(x,y,z,u,v), g(x,y,z,u,v)$ is polynomial, it is $\mathcal{C}^1$. The differential of F is

$$dF(x,y,z,u,v) = \begin{pmatrix} 1 & 2 & -v & 3 & -z \\ \\ y & x & u & z & 1 \end{pmatrix}.$$

The determinant of the last 2×2 submatrix is $3 + z^2 > 0$, so it is invertible: the linearization of F at a point $(x_0, y_0, z_0, u_0, v_0) \in S$ may be solved for (u, v) as functions of (x, y, z). By the implicit function theorem, there is an open set $U \in \mathbf{R}^5$ such that $(x_0, y_0, z_0, u_0, v_0) \in U$ and an open set $V \in \mathbf{R}^3$ such that $(x_0, y_0, z_0) \in V$ and $\mathcal{C}^1$ functions $h(x,y,z)$ and $k(x,y,z)$ on V such that

$$(x,y,z,u,v) \in U \cap S \qquad \iff \qquad u = h(x,y,z) \text{ and } v = k(x,y,z) \text{ for some } (x,y,z) \in V.$$

It follows that S is a parameterized $p = 3$ dimensional surface as it is a graph. The function

$$G(x,y,z) = (x, y, z, h(x,y,z), k(x,y,z))$$

parameterizes the surface near $(x_0, y_0, z_0, u_0, v_0)$. It is one-to-one since it is a graph and dG has rank three because $dG_{x,y,z} = I$.

4. *Determine whether the statement is true or false. If true give a brief reason. If false, give a counterexample.*

   (a) Statement. *Let $U \subset \mathbf{R}^d$ be open and $K \subset U$ be a compact convex set with nonempty interior and $f \in \mathcal{C}^1(U)$. Then there is $L < \infty$ such that $|f(x) - f(y)| \le L|x - y|$ for all $x, y \in K$.*

   True. Since f is continuously differentiable, $df(x)$ is continuous on a compact set K so that

$$L = \sup_{x \in K} \|df(x)\|$$

is finite and bounds the gradient. Since K is convex, for every pair of points $x, y \in K$ the whole line segment $\overline{xy}$ is in K. Thus, by the Mean Value Theorem, there is $c \in \overline{xy} \subset K$ such that

$$f(y) - f(x) = df(c)(y - x)$$

so that

$$|f(y) - f(x)| \le \|df(c)\| \, \|y - x\| \le L\|y - x\|.$$

(b) STATEMENT. *Let $E \subset \mathbf{R}^2$ consist of countably infinitely many distinct points $E = \bigcup_{i=1}^{\infty} \{x_i\}$. Then E is not a Jordan region.*

FALSE. (BE CAREFUL ABOUT LOGIC!) For some collections of points, E may be a Jordan region. For example, the set

$$E = \left\{ \left(\frac{1}{n}, \frac{1}{n} \right) : n \in \mathbb{N} \right\}$$

is contained in the segment $S = \{(t,t) : 0 \le t \le 1\}$, which has volume zero in $\mathbf{R}^2$. Thus, as its subset, E also has volume zero and is a Jordan region.

Of course, some subsets are not Jordan regions. For example if $E' = \{x_n\}$ is an enumeration of the rational points in the unit square $[0,1] \times [0,1]$ then $\partial E'$ is the whole square of positive volume, so this E' is not a Jordan region.

(c) STATEMENT. *Let $R = [0,1]^2$, $f, f_n : R \to \mathbf{R}$ be integrable functions such that $f(x,y) = \lim_{n\to\infty} f_n(x,y)$ for all $(x,y) \in R$. Then $\int_R f(x,y)\, dV(x,y) = \lim_{n\to\infty} \int_R f_n(x,y)\, dV(x,y)$.*

FALSE. The function

$$f_n(x,y) = \begin{cases} n^2, & \text{if } 0 < x < \frac{1}{n} \text{ and } 0 < y < \frac{1}{n}; \\ 0, & \text{otherwise.} \end{cases}$$

converges pointwise to $f = 0$. But, since f_n are continuous except on a set of volume zero, they are integrable. Moreover $\int_R f_n(x,y)\, dV(x,y) = 1$ for all n, which does not converge to zero.

5. *Let $R \subseteq \mathbf{R}^d$ be an be an aligned rectangle in the in the plane and f be a real valued function on R. Define both: f is integrable on R and $\int_R f(\mathbf{x})\, dV(\mathbf{x})$. Complete the statement of a theorem. Using only your theorem, show that a linear function $f(x) = ax + by$, where a and b are constants, is integrable on $R = [0,2] \times [0,3]$.*

Theorem. *Let $R \subset \mathbf{R}^d$ be a rectangle and $f : R \to \mathbf{R}$ be a bounded function. Then f is integrable on R if and only if*

There is a sequence of partitions $\mathcal{P}_n$ of R such that

$$U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) \to 0 \quad \text{as } n \to \infty.$$

f is *integrable in R* means that the upper integral equals the lower integral

$$\underline{\int}_R f(x)\, dV(x) = \overline{\int}_R f(x)\, dV(x)$$

where

$$\underline{\int}_R f(x)\,dV(x) = \sup_{\mathcal{P}} L(f,\mathcal{P}), \qquad \overline{\int}_R f(x)\,dV(x) = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

where inf and sup of lower and upper sums are taken over partitions of R. If f is integrable on R, then its integral is the common value

$$\int_R f(x)\,dV(x) = \underline{\int}_R f(x)\,dV(x) = \overline{\int}_R f(x)\,dV(x).$$

Let $\mathcal{P}_n = \{S_i\}$ be the partition of R into $6n^2$ many $\frac{1}{n} \times \frac{1}{n}$ squares S_i. f is $\mathcal{C}^1$ since it is linear and its gradient $df(x,y) = (a,b)$ has norm $L = \sqrt{a^2+b^2}$. By Problem 4a above,

$$|f(x,y) - f(x_i,y_i)| \le L\|(x,y) - (x_i,y_i)\|$$

where $(x,y) \in S_i$ is any point and (x_i,y_i) is in the center of S_i. It follows that

$$M_i = \sup_{(x,y)\in S_i} f(x,y) \le f(x_i,y_i) + \sup_{(x,y)\in S_i} \big[f(x,y) - f(x_i,y_i)\big] \le f(x_i,y_i) + \frac{L\,\mathrm{diam}(S_i)}{2}$$
$$m_i = \inf_{(x,y)\in S_i} f(x,y) \ge f(x_i,y_i) + \inf_{(x,y)\in S_i} \big[f(x,y) - f(x_i,y_i)\big] \ge f(x_i,y_i) - \frac{L\,\mathrm{diam}(S_i)}{2}.$$

Since the diameter of S_i is $\frac{\sqrt{2}}{n}$, it follows that

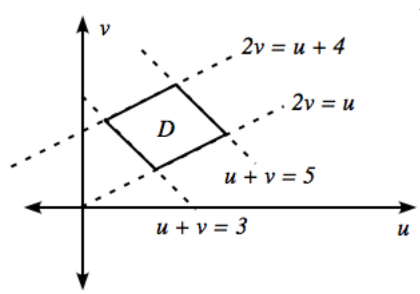
$$M_i - m_i \le L\,\mathrm{diam}(S_i) = \frac{\sqrt{2}L}{n}.$$

Summing over all squares of the partition,

$$U(f,\mathcal{P}_n) - L(f,\mathcal{P}_n) = \sum_{i=1}^{6n^2} (M_i - m_i)V(S_i) \le 6n^2 \cdot \frac{\sqrt{2}L}{n} \cdot \frac{1}{n^2} \to 0$$

as $n \to \infty$. Thus f is integrable on R by the theorem.

6. *Let $D \subseteq \mathbf{R}^2$ be the region in the first quadrant bounded by the curves $u+v=3$, $u+v=5$, $2v=u$, and $2v=u+4$. Find an open set $U \subseteq \mathbf{R}^2$, a one-to-one function $\varphi \in \mathcal{C}^1(U,\mathbf{R}^2)$ such that $\det(\mathrm{d}\varphi(x,y)) \neq 0$ for all $(x,y) \in U$ and an aligned rectangle $R \subseteq U$ such that $D = \varphi(R)$. By changing variables using φ, find the integral $\int_D u\,dV(u,v)$.*



Use the boundary curves to make the transformation

$$x = u + v$$
$$y = -u + 2v$$

Then $R = \{(x,y) : 3 \le x \le 5 \text{ and } 0 \le y \le 4\}$. Solving for (u,v) in terms of (x,y) we find

$$\begin{pmatrix} u \\ \\ v \end{pmatrix} = \varphi \begin{pmatrix} x \\ \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{3}x - \frac{1}{3}y \\ \\ \frac{1}{3}x + \frac{1}{3}y \end{pmatrix}$$

The differential

$$d\varphi = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

whose determinant is $\frac{1}{9}$ so $d\varphi(x,y)$ is invertible. Thus φ is one to one because the linear map is invertible, $\mathcal{C}^1$ since it's linear, with nonsingular Jacobian in the whole plane $U = \mathbf{R}^2$ such that $D = \varphi(R)$.

Because also the function $f(u,v) = u$ is continuous on U, we satisfy conditions for the change of variables formula and iterated integrals

$$\begin{aligned}
\int_D u\, dV(u,v) &= \int_{\varphi(R)} f(u,v)\, dV(u,v) \\
&= \int_R f(\varphi(x,y))\, |\det d\varphi(x,y)|\, dV(x,y) \\
&= \int_R \left(\frac{2}{3}x - \frac{1}{3}y\right) \cdot \frac{1}{9}\, dV(x,y) \\
&= \frac{1}{27} \int_{x=3}^{5} \int_{y=0}^{4} (2x - y)\, dy\, dx \\
&= \frac{1}{27} \int_{x=3}^{5} \left[2xy - \frac{y^2}{2}\right]_{y=0}^{4} dx \\
&= \frac{1}{27} \int_{x=3}^{5} [8x - 8]\ dx \\
&= \frac{1}{27} \left[4x^2 - 8x\right]_{x=3}^{5} \\
&= \frac{1}{27} [4 \cdot 16 - 8 \cdot 2] = \frac{16}{9}.
\end{aligned}$$

7. (a) *Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by* $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \sin y \\ \sqrt{1+x^2+y^2} \end{pmatrix}$. *Determine whether f is uniformly continuous on $\mathbf{R}^2$ and prove your result.*

f is UNIFORMLY CONTINUOUS on $\mathbf{R}^2$. Choose $\epsilon > 0$. Let $\delta = \frac{\epsilon}{6}$. Choose $(x_1, y_1), (x_2, y_2) \in \mathbf{R}^2$ such that $\|(x_1, y_1) - (x_2, y_2)\| < \delta$, we find using $\|(u, v)\| \le |u| + |v|$ and $|\sin u - \sin v| \le |u - v|$ for all u, v,

$$\begin{aligned}
\|f(x_1,y_1)-f(x_2,y_2)\| &\le |(x_1+\sin y_1)-(x_2+\sin y_2)| + \left|\sqrt{1+x_1^2+y_1^2}-\sqrt{1+x_2^2+y_2^2}\right| \\
&\le |x_1-x_2| + |\sin y_1 - \sin y_2| + \frac{|(1+x_1^2+y_1^2)-(1+x_2^2+y_2^2)|}{\sqrt{1+x_1^2+y_1^2}+\sqrt{1+x_2^2+y_2^2}} \\
&\le |x_1-x_2| + |y_1-y_2| + \frac{|x_1^2-x_2^2|+|y_1^2-y_2^2|}{\sqrt{1+x_1^2+y_1^2}+\sqrt{1+x_2^2+y_2^2}} \\
&= |x_1-x_2| + |y_1-y_2| + \frac{|x_1+x_2|\,|x_1-x_2|+|y_1+y_2|\,|y_1-y_2|}{\sqrt{1+x_1^2+y_1^2}+\sqrt{1+x_2^2+y_2^2}} \\
&\le |x_1-x_2| + |y_1-y_2| + \frac{(|x_1|+|x_2|)\,|x_1-x_2|+(|y_1|+|y_2|)\,|y_1-y_2|}{\sqrt{1+x_1^2+y_1^2}+\sqrt{1+x_2^2+y_2^2}} \\
&\le \left(1+\frac{|x_1|}{\sqrt{1+x_1^2+y_1^2}}+\frac{|x_2|}{\sqrt{1+x_2^2+y_2^2}}\right)|x_1-x_2| \\
&\quad + \left(1+\frac{|y_1|}{\sqrt{1+x_1^2+y_1^2}}+\frac{|y_2|}{\sqrt{1+x_2^2+y_2^2}}\right)|y_1-y_2|.
\end{aligned}$$

Thus using $|u| \le \|(u, v)\|$ and $|v| \le \|(u, v)\|$,

$$\begin{aligned}
\|f(x_1,y_1)-f(x_2,y_2)\| &\le 3|x_1-x_2| + 3|y_1-y_2| \\
&\le 6\|(x_1-y_1, x_2-y_2)\| \\
&= 6\|(x_1,x_2)-(y_1,y_2)\| < 6\delta = \epsilon.
\end{aligned}$$

Hence, f is uniformly continuous on $\mathbf{R}^2$. Indeed, it is Lipschitz on $\mathbf{R}^2$.

(b) *Let $f : \mathbf{R}^d \to \mathbf{R}$ be continuous. Show that $E = \{x \in \mathbf{R}^d : f(x) > 0\}$ is open.*

To show that E is open, we have to show that for every $z \in E$ there is $\delta > 0$ such that the whole open ball $B_\delta(z) \subset E$. Choose $z \in E$. Then $f(z) > 0$. By continuity, we argue that $f(x) > 0$ for every x close to z. Indeed, let $\epsilon = f(z) > 0$. By continuity of f, there is a $\delta > 0$ such that $|f(x) - f(z)| < \epsilon$ whenever $\|x - z\| < \delta$. I claim that if $x \in B_\delta(z)$ then $f(x) > 0$ so $x \in E$. In other words, $B_\delta(z) \subset E$ so E is open.

To see the claim, suppose $x \in B_\delta(z)$. Hence $\|z - x\| < \delta$. Thus

$$f(x) = f(z) - (f(z) - f(x)) \ge f(z) - |f(z) - f(x)| > f(z) - \epsilon = f(z) - f(z) = 0.$$

8. *Define:* $E \subseteq \mathbf{R}^n$ *is a* Jordan Region. *Show that E is a Jordan region, where*

$$E = \{(x,y) \in \mathbf{R}^2 : \ (x = 0 \text{ and } 0 \le y \le 1) \text{ or } (y = 0 \text{ and } 0 \le x \le 1) \ \}.$$

A set $E \in \mathbf{R}^n$ is a *Jordan Region* if the characteristic function

$$\chi_E = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \notin E. \end{cases}$$

is integrable in any aligned rectangle $R \subset \mathbf{R}^n$ such that $E \subset R$. Integrable in R means that the upper integral equals the lower integral

$$\underline{\int}_R \chi_E(x)\, dV(x) = \overline{\int}_R \chi_E(x)\, dV(x)$$

where

$$\underline{\int}_R \chi_E(x)\, dV(x) = \sup_{\mathcal{P}} L(\chi_E, \mathcal{P}), \qquad \overline{\int}_R \chi_E(x)\, dV(x) = \inf_{\mathcal{P}} U(\chi_E, \mathcal{P})$$

where inf and sup of lower and upper sums are taken over partitions of R.

The set E is "L" shaped. We show that the characteristic function χ_E is integrable in $R = [0,1] \times [0,1]$. We use the theorem in Problem 5 above as the criterion of integrability. Let the partition $\mathcal{P}_n = \{S_i\}$ divide R into n^2 many $\frac{1}{n} \times \frac{1}{n}$ subsquares S_i. Then $M_i = \sup\{\chi_E(x,y) : (x,y) \in S_i\} = 1$ for all the squares S_i along the $x = 0$ and $y = 0$ edges of the square. There are $2n - 1$ such subsquares S_i that touch E. All other $M_i = 0$. Also $m_i = \inf\{\chi_E(x,y) : (x,y) \in S_i\} = 0$ for all the squares S_i . Thus

$$\begin{aligned} U(\chi_E, \mathcal{P}_n) - L(\chi_E, \mathcal{P}_n) &= \sum_{i=1}^{n^2} (M_i - m_i) V(S_i) \\ &= \sum_{S_i \cap E \ne \emptyset} (1 - 0) \frac{1}{n^2} \\ &= \frac{2n - 1}{n^2} \to 0 \end{aligned}$$

as $n \to \infty$. Hence, χ_E is integrable so E is a Jordan Region. Because $U(\chi_E, \mathcal{P}_n) - L(\chi_E, \mathcal{P}_n) = U(\chi_E, \mathcal{P}_n)$, this shows that $0 \le \bar{V}(E) \le U(\chi_E, \mathcal{P}_n) \to 0$ as $n \to \infty$ so E is a Jordan Region of volume zero.