Math 3210 § 2. **First Midterm Exam** Name: Solutions
Treibergs February 2, 2017

1. *Let $x > -1$. Prove that for every integer $n \geq 0$,*

$$\mathcal{P}(n) : (1+x)^n \geq 1 + nx.$$

We use induction starting at $n = 0$ which works just as well as starting from $n = 1$. Note that $x > -1$ implies that $1 + x > 0$. Hence in the base case, $n = 0$, $(1 + x)^0 = 1 = 1 + 0x$ so $\mathcal{P}(0)$ holds.

For the induction case, assume that for some $n \geq 0$, $\mathcal{P}(n)$ holds to show $\mathcal{P}(n+1)$ holds. The induction hypothesis $P(n)$ says

$$(1+x)^n \geq 1 + nx.$$

But since $(1 + x) > 0$, we preserve the order when we multiply the inequality. This gives the induction step

$$(x+1)^{n+1} = (x+1)(x+1)^n \geq (1+x)(1+nx) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x$$

because $nx^2 \geq 0$.

2. *Recall the axioms of a commutative ring $(R, +, X)$. For any $x, y, z \in R$,*

| | | |
|---|---|---|
| [A1.] | (Commutativity of Addition) | $x + y = y + x.$ |
| [A2.] | (Associativity of Addition) | $x + (y + z) = (x + y) + z.$ |
| [A3.] | (Additive Identity.) | $(\exists\, 0 \in R)\,(\forall\, t \in R)\ 0 + t = t.$ |
| [A4.] | (Additive Inverse) | $(\exists -x \in R)\ x + (-x) = 0.$ |
| [M1.] | (Commutativity of Multiplication) | $xy = yx.$ |
| [M2.] | (Associativity of Multiplication) | $x(yz) = (xy)z.$ |
| [M3.] | (Multiplicative Identity.) | $(\exists\, 1 \in R)\ 1 \neq 0$ and $(\forall\, t \in R)\ 1t = t.$ |
| [D.] | (Distributivity) | $x(y + z) = xy + xz.$ |

*Using only the axioms of a commutative ring, show that for any $a, b \in R$, then the equation*

$$a + x = b$$

*has a unique solution $x = (-a) + b$. Justify every step of your argument using just the axioms listed here.*

First we show $x = (-a) + b$ solves the equation.

| | |
|---|---|
| $a + x = a + ((-a) + b))$ | Substitute $x$. |
| $= (a + (-a)) + b$ | Associativity of addition, A2. |
| $= 0 + b$ | Additive inverse A4. |
| $= b.$ | Additive identity A3. |

Second we argue the solution is unique. Suppose $x$ and $z$ were two solutions. Then both satisfy the equation

$$\begin{aligned}
a + x &= b \\
a + z &= b && \text{Substitute solutions } x \text{ and } z. \\
a + x &= a + x && \text{Both equal } b. \\
(-a) + (a + x) &= (-a) + (a + z) && \text{Pre-add } -a \text{ (which exists by A4) to both sides.} \\
((-a) + a) + x &= ((-a) + a) + z && \text{Associativity of addition A2.} \\
(a + (-a)) + x &= (a + (-a)) + z && \text{Commutativity of addition A1.} \\
0 + x &= 0 + z && \text{Additive inverse A4.} \\
x &= z && \text{Additive identity A3.}
\end{aligned}$$

Thus any two solutions are the same.

Another argument may be given. We start from the equation and deduce the value of the unknown.

$$\begin{aligned}
a + x &= b && \text{Given.} \\
(-a) + (a + x) &= (-a) + b && \text{Pre-add } -a \text{ (which exists by A4) to both sides.} \\
((-a) + a) + x &= (-a) + b && \text{Associativity of addition, A2.} \\
(a + (-a)) + x &= (-a) + b && \text{Commutitvity of addition, A1.} \\
0 + x &= (-a) + b && \text{Additive inverse, A4.} \\
x &= (-a) + b && \text{Additive identity, A3.}
\end{aligned}$$

Thus we deduce that the equation may be solved by the number $x = (-a) + b$. This argument says more. No matter which solution $x$ was used, the argument showed that all solutions are the same one and only solution $x = (-a) + b$. Hence the solution is unique.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *If $f : A \to B$ then $f(A \backslash E) = f(A) \backslash f(E)$ for every subset $E \subset A$.*
   FALSE. Let $A = B = \mathbb{R}$, $E = [0, \infty)$, $f(x) = x^2$ (which is not one-to-one), $A \backslash E = (-\infty, 0)$, $f(A \backslash E) = (0, \infty)$, $f(A) = f(E) = [0, \infty)$ so $f(A) \backslash f(E) = \emptyset \neq f(A \backslash E)$.

   (b) *Let $f : X \to Y$. If $f^{-1}(E) = X$ for some proper subset $E$ of $Y$ then $f$ is not onto.*
   TRUE. If $E \subset Y$ is a proper subset, it is not all of $Y$ so there is $y_0 \in Y$ but $y_0 \notin E$. Since the range $f(X) = E$, no point of $X$ maps to $y_0$, so $f$ is not onto.

   (c) *Let $f : X \to Y$ be a function. Suppose that for every $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$ implies $x_1 \neq x_2$. Then $f$ is one-to-one.*
   FALSE. The statement is true for every function. *e.g.*, $g(x) = x^2$ is not one-to-one on $\mathbb{R}$, but the hypothesis is true as can be seen by its contrapositive: $x_1 = x_2$ implies $x_1^2 = g(x_1) = g(x_2) = x_2^2$.

4. *Recall that the rational numbers are defined to be the set of equivalence classes* $\mathbb{Q} = S/\sim$ *where* $S = \left\{\frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0\right\}$ *is the set of symbols (pairs of integers) and the symbols are equivalent if they represent the same fraction* $\frac{a}{b} \sim \frac{c}{d}$ *iff* $ad = bc$*. We denote the equivalence class, the "fraction,"* $\left[\frac{a}{b}\right]$ *to distinguish it from a symbol from* $S$*. Multiplication, for example is defined on equivalence classes by* $\left[\frac{m}{n}\right] \cdot \left[\frac{r}{t}\right] = \left[\frac{(mr)}{(nt)}\right]$.

   (a) *Given fractions* $x = \left[\frac{m}{n}\right]$, $y = \left[\frac{r}{t}\right]$ *in* $\mathbb{Q}$*, suppose we define the operation*

$$x \ominus y := \left[\frac{mt - nr}{nt}\right].$$

   *Show that the definition of* $\ominus$ *is well defined: it does not depend on the choice of the symbols representing the fractions.*

   Let $\frac{m'}{n'} \sim \frac{m}{n}$ so $m'n = mn'$ and $\frac{r'}{t'} \sim \frac{r}{t}$ so $r't = rt'$. Then we claim that the formulae are equivalent: $\frac{m't' - n'r'}{n't'} \sim \frac{mt - nr}{nt}$. To see this, using $m'n = mn'$ and $r't = rt'$,

$$nt(m't' - n'r') = tt'nm' - nn'tr' = tt'n'm - nn't'r = n't'(mt - nr).$$

   Thus $\frac{m't' - n'r'}{n't'} \sim \frac{mt - nr}{nt}$.

   (b) *Define the subset* $\mathcal{P} = \left\{\left[\frac{p}{q}\right] \in \mathbb{Q} : p \geq 0 \text{ and } q > 0.\right\}$*. An order is defined on* $\mathbb{Q}$ *by* $x \preceq y$ *iff* $y \ominus x \in \mathcal{P}$*. Show that with this "*$\preceq$*," the rationals* $\mathbb{Q}$ *satisfy the order axiom O1: For all* $x, y \in \mathbb{Q}$*, either* $x \preceq y$ *or* $y \preceq x$*.*

   Let $x = \left[\frac{m}{n}\right]$, $y = \left[\frac{r}{t}\right]$. Then $x \ominus y := \left[\frac{mt - nr}{nt}\right]$ and $y \ominus x := \left[\frac{nr - mt}{nt}\right]$. Notice that the numerators are negatives: $-(mt - nr) = nr - mt$ so that by the order properties of $\mathbb{Z}$, one or the other is nonnegative (or both are zero). So if $nt > 0$, one or the other $x \ominus y$ or $y \ominus x$ is in $\mathcal{P}$. On the other hand, if $nt < 0$, we may choose an equivalent representative $x = \left[\frac{-m}{-n}\right]$. We have $\frac{-m}{-n} \sim \frac{m}{n}$ because $n(-m) = (-n)m$. Now computing using the new $x$, $x \ominus y := \left[\frac{(-m)t - (-n)r}{(-n)t}\right]$ and $y \ominus x := \left[\frac{(-n)r - (-m)t}{(-n)t}\right]$. Now the denominator is positive $(-n)t > 0$ and the numerators are still negatives of one another, so one of them has to be nonnegative, thus, again, $x \ominus y$ or $y \ominus x$ is in $\mathcal{P}$.

5. *Let $E \subset \mathbb{R}$ be a set of real numbers given by*

$$E = \{x \in \mathbb{R} : \quad (\forall \zeta \in \mathbb{Z}) \quad (\exists \tau > 0) \quad (\tau \le |x - \zeta|) \quad \}.$$

*Find $E$ and and prove your result.*

$$\begin{aligned} E &= \{x \in \mathbb{R} : \quad (\forall \zeta \in \mathbb{Z}) \quad (\exists \tau > 0) \quad (\tau \le |x - \zeta|) \quad \} \\ &= \bigcap_{\zeta \in \mathbb{Z}} \bigcup_{\tau > 0} \Big\{ (-\infty, z - \tau] \cup [z + \tau, \infty) \Big\} \\ &= \bigcap_{\zeta \in \mathbb{Z}} \Big\{ (-\infty, z) \cup (z, \infty) \Big\} \\ &= \mathbb{R} \backslash \mathbb{Z}. \end{aligned}$$

To prove it, we show that the complement $E^c = \mathbb{Z}$.

To show "$\subset$," choose $x \in E^c$ to show $x \in \mathbb{Z}$.

$$E^c = \{x \in \mathbb{R} : \quad (\exists \zeta \in \mathbb{Z}) \quad (\forall \tau > 0) \quad (\tau > |x - \zeta|) \quad \}$$

Let $\zeta_0 \in \mathbb{Z}$ correspond to $x$. Then $x$ satisfies

$$(\forall \tau > 0) \quad (\tau > |x - \zeta_0|).$$

In other words, $x = \zeta_0$ which is an integer, so $x \in \mathbb{Z}$.

To show "$\supset$," choose $x \in \mathbb{Z}$ to show $x \in E^c$. Take $\zeta = x$. Then for all $\tau > 0$ we have $\tau > |x - \zeta| = 0$ so $x$ satisfies the condition to be in $E^c$. Hence we have shown $E^c = \mathbb{Z}$ so $E = \mathbb{R} \backslash \mathbb{Z}$.