Math 3210 § 1. **First Midterm Exam** Name: Solutions
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1. *The Fibonacci Sequence is defined recursively. Prove that* $f_n \le \varphi^n$ *for all* $n \in \mathbf{N}$, *where* $\varphi = \dfrac{1+\sqrt{5}}{2}$.

$$f_1 = 1, \qquad f_2 = 1, \qquad \text{and} \qquad f_{n+1} = f_n + f_{n-1} \text{ for all } n \ge 2.$$

We prove the statement using mathematical induction. For each $n \in \mathbf{N}$ we have the statement

$$\mathcal{P}_n = \text{“ } f_n \le \varphi^n \text{ and } f_{n-1} \le \varphi^{n-1}.\text{”}$$

We could have also used strong mathematical induction that assumes the truth of all previous statements as its hypothesis.

Base Case. When $n = 2$, $f_1 = 1 \le \varphi$ and $f_2 = 1 \le \varphi^2 = \dfrac{3+\sqrt{5}}{2}$.

Induction case. For any $n \ge 2$ we assume that $\mathcal{P}_n$ is true. Thus we assume $f_n \le \varphi^n$ which is the second half of $\mathcal{P}_{n+1}$. To verify the other half, observe that by the recursion formula and induction hypothesis,

$$f_{n+1} = f_n + f_{n-1} \le \varphi^n + \varphi^{n-1} = (\varphi + 1)\varphi^{n-1} = \varphi^2 \cdot \varphi^{n-1} = \varphi^{n+1},$$

where we used the fact that $\varphi + 1 = \varphi^2$. Thus the induction step is complete.

Since both the base and inductions cases hold, $\mathcal{P}_n$ is true for all $n \ge 2$, namely $f_n \le \varphi^n$ for all $n \in \mathbf{N}$.

2. *Recall the axioms of a field F with operations $+$ and $\times$: For any $x, y, z \in F$,*

A1.	*Commutativity of Addition*	$x + y = y + x.$
A2.	*Associativity of Addition*	$x + (y + z) = (x + y) + z.$
A3.	*Additive Identity*	$(\exists\, 0 \in F)\, (\forall\, t \in F)\ 0 + t = t.$
A4.	*Additive Inverse*	$(\exists\, {-x} \in F)\ x + (-x) = 0.$
M1.	*Commutativity of Multiplication*	$xy = yx.$
M2.	*Associativity of Multiplication*	$x(yz) = (xy)z.$
M3.	*Multiplicative Identity*	$(\exists\, 1 \in F)\ 1 \neq 0$ *and* $(\forall\, t \in F)\ 1t = t.$
M4.	*Multiplicative Inverse*	*If* $x \neq 0$ *then* $(\exists\, x^{-1} \in F)\ x^{-1}x = 1.$
D.	*Distributivity*	$x(y + z) = xy + xz.$

Using only the axioms of a field, show that if $a, b \in F$ such that $a \neq 0$ and $b \neq 0$ then $a^{-1} + b^{-1} = (a + b)(a^{-1}b^{-1})$. Justify every step of your argument using just the axioms listed here. [Hint: the first line of your proof must not be "$a^{-1} + b^{-1} = (a + b)(a^{-1}b^{-1})$."]

$a^{-1} + b^{-1} = 1 \cdot a^{-1} + 1 \cdot b^{-1}$	Multiplicative Identity M3.
$= a^{-1} \cdot 1 + b^{-1} \cdot 1$	Commutativity of Multiplication M1.
$= a^{-1}(b^{-1}b) + b^{-1}(a^{-1}a)$	Since $a, b \neq 0$ use Multiplicative Inverses M4.
$= (a^{-1}b^{-1})\, b + (b^{-1}a^{-1})\, a$	Associativity of Multiplication M2.
$= (a^{-1}b^{-1})\, b + (a^{-1}b^{-1})\, a$	Commutativity of Multiplication M1.
$= (a^{-1}b^{-1})(b + a)$	Distributivity D.
$= (b + a)(a^{-1}b^{-1})$	Commutativity of Multiplication M1.
$= (a + b)(a^{-1}b^{-1})$	Commutativity of Addition A1.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT. *Let $f : \mathbf{R} \to \mathbf{R}$ be a function. If f is not one-to-one then f is not onto.*
   FALSE. The function $f(x) = x^3 - x$ is onto (its graph crosses every horizontal line) but not one-to-one since $f(0) = 0 = f(1)$.

   (b) STATEMENT. *Let $f : \mathbf{A} \to \mathbf{B}$ be a function. Then $f(E \backslash F) = f(E) \backslash f(F)$ for all subsets $E, F \subset A$.*
   FALSE. Define $A = \{1, 2\}$, $B = \{3\}$, $E = \{1\}$, $F = \{2\}$ and $f(1) = f(2) = 3$. Then $E \backslash F = E$ so $f(E \backslash F) = \{3\}$ which is not equal to $f(E) \backslash f(F) = \{3\} \backslash \{3\} = \emptyset$.

   (c) STATEMENT. $(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})(\forall z \in \mathbf{R})(\ x + z > y + z\ )$.
   TRUE. Here is the proof: choose $x \in \mathbf{R}$. Let $y = x - 1$. Then for any $z \in \mathbf{R}$ we have $x + z > x - 1 + z = y + z$.

4. *State the definition: The function $f : A \to B$ is* one-to-one. *Let $f : A \to B$ be a one-to-one function. Show that $E = f^{-1}\big(f(E)\big)$ for all subsets $E \subset A$.*

   Assume that $E \subset A$ is any subset. We wish to show first $E \subset f^{-1}\big(f(E)\big)$ and second $E \supset f^{-1}\big(f(E)\big)$.

   First choose $x \in E$ to show $x \in f^{-1}\big(f(E)\big)$. $x \in E$ implies that $f(x) \in f(E) = S$. But from the meaning preimage this says $x \in f^{-1}(S)$ so we have that $x \in f^{-1}(f(E))$.

   Second choose $x \in f^{-1}\big(f(E)\big)$ to show $x \in E$. $x \in f^{-1}\big(f(E)\big)$ implies that $y = f(x) \in f(E)$ by meaning of preimage. Now $y \in f(E)$ implies that there is $z \in E$ such that $f(z) = y = f(x)$. However we have assumed that f is one-to-one, which implies that $x = z$. Thus we have shown that $x = z \in E$, completing the proof.

5. *Let $E \subset \mathbf{R}$ be a nonempty subset which is bounded above. Define the* least upper bound*: $L = \operatorname{lub} E$. Find $L = \operatorname{lub} E$ if it exists, and prove your answer where*

$$E = \left\{ \frac{p}{q} : \ p, q \in \mathbf{N} \text{ such that } p < 2q \right\}$$

   The least upper bound of a set is a number L that is first, an upper bound: for every $x \in E$ we have $x \le L$. Second, L is least among upper bounds, or to put it another way, no smaller number can be an upper bound: if $M < L$ then there is $x \in E$ such that $M < x$.

   We show that $\operatorname{lub} E = 2$. First we argue that $L = 2$ is an upper bound. Indeed, for any $\frac{p}{q} \in E$ then $p, q \in \mathbf{N}$ such that $p < 2q$. But this implies that $\frac{p}{q} < 2$, so $L = 2$ is an upper bound.

   Second, suppose that $M < 2$ is a smaller number. By the Archimedean Property, there is $q \in \mathbf{N}$ so that $\frac{1}{q} < 2 - M$. Put $p = 2q - 1$. Since p is an integer such that $p = 2q - 1 \ge 2 \cdot 1 - 1 = 1$ we have $p \in \mathbf{N}$. Since $p = 2q - 1 < 2q$ we have that $\frac{p}{q} \in E$. On the other hand,

$$\frac{p}{q} = \frac{2q-1}{q} = 2 - \frac{1}{q} > 2 - (2 - M) = M.$$

   Thus we have shown that M cannot be a lower bound: there is $\frac{p}{q} \in E$ such that $M < \frac{p}{q}$. $\square$

   An alternative argument might involve the density of rationals to provide a rational number $\frac{p}{q}$ in the interval $(M, 2)$.