Math 3210 § 3. **Third Midterm Exam** Name: Solutions
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1. *Let $\mathcal{D} \subset \mathbf{R}$ be a nonempty set and $f_n, f : \mathcal{D} \to \mathbf{R}$ be functions. Define: $\{f_n\}$ converges uniformly on $\mathcal{D}$ to a function f. Find the limiting function $f(x)$ and prove that the sequence $f_n(x) = \dfrac{1}{1+nx}$ converges pointwise to $f(x)$ on $[0,\infty)$. Determine whether the convergence is uniform and prove your result.*

   A sequence of functions $\{f_n\}$ is said to *converge uniformly* on $\mathcal{D}$ to a function f if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that
   $$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever } x \in \mathcal{D} \text{ and } n > N.$$
   Note that $f_n(0) = 1$ for every n so that $f_n(0) \to 1$ as $n \to \infty$. On the other hand, if $x > 0$, then by the Main Limit Theorem,
   $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n} + x} = \frac{0}{0+x} = 0.$$
   Thus we have shown that $f_n(x) \to f(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x > 0. \end{cases}$ pointwise on $[0,\infty)$.

   This convergence is Not Uniform. We verify the definition that the convergence $f_n \to f$ is not uniform on $[0,\infty)$. Let $\varepsilon = \frac{1}{2}$. Choose $N \in \mathbf{R}$. By the Archimedean Property, there is $n \in \mathbf{N}$ such that $n > N$. Let $x_n = \frac{1}{n}$. Then for these n and $x_n \in [0,\infty)$ we have
   $$|f_n(x_n) - f(x_n)| = \left| \frac{1}{1+nx_n} - 0 \right| = \frac{1}{1+1} \ge \varepsilon.$$
   Alternately we could have observed that that the discontinuous f could not have been the uniform limit of the contnuous f_n's. Or we may have observed that $\{x_n\}$ is a sequence in $\mathcal{D}$ such that $|f_n(x_n) - f(x_n)|$ does not converge to zero as it must do for every sequence when the convergence is uniform.

2. *Suppose that a function $f : \mathbf{R} \to \mathbf{R}$ is continuous at $a \in \mathbf{R}$ and that $f(x) > 0$ for all $x \ne a$. Prove that $f(a) \ge 0$.*

   Choose $\varepsilon > 0$. By the continuity of f at a, there is a $\delta > 0$ such that
   $$|f(x) - f(a)| < \varepsilon \qquad \text{whenever } x \in \mathbf{R} \text{ such that } |x - a| < \delta.$$
   Pick such x, say, $x_0 = a + \delta/2$. Then for this x_0, since $a < x_0 < a + \delta$ and $f(x_0) > 0$ we have
   $$f(a) = f(x_0) + f(a) - f(x_0) \ge f(x_0) - |f(a) - f(x_0)| > 0 - \varepsilon = -\varepsilon.$$
   But since $\varepsilon > 0$ is arbitrary, we conclude that $f(a) \ge 0$.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) Statement *Let $f : [0,1] \to \mathbf{R}$. If $f(\,[0,1]\,)$ is a closed and bounded interval, then f is continuous.*

   False. If we knew that f were strictly monotone then the conclusion follows. Without monotonicity we construct a counterexample. Take
   $$f(x) = \begin{cases} x, & \text{if } x \le \frac{1}{2}; \\ x - \frac{1}{2}, & \text{if } x > \frac{1}{2}. \end{cases}$$
   Then f is not continuous at $x = \frac{1}{2} \in [0,1]$ but $f(\,[0,1]\,) = [0, \frac{1}{2}]$.

(b) STATEMENT *There is a point $x \in \mathbf{R}$ such that* $f(x) = \dfrac{1+x+x^2+x^3}{1+x+x^2} = 2.$

TRUE. Since $f(x)$ is a rational function whose denominator doesn't vanish because

$$1+x+x^2 = \frac{3}{4} + \left(\frac{1}{2}+x\right)^2 \ge \frac{3}{4}$$

we know that f is continuous on $\mathbf{R}$. Because $f(0) = \dfrac{1}{1} = 1$ and $f(2) = \dfrac{1+2+4+8}{1+2+4} = \dfrac{15}{7} > 2$ we see that $y = 2$ is between $f(0)$ and $f(2)$. It follows from the Intermediate Value Theorem that there is $c \in [0,2]$ such that $f(c) = 2$.

(c) STATEMENT *Suppose the real sequence $\{a_n\}$ satisfies* $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. *Then $\{a_n\}$ is convergent.*

FALSE. Consider the sequence $a_n = \sqrt{n}$. Then $|a_{n+1} - a_n| =$

$$= |\sqrt{n+1} - \sqrt{n}| = \left|\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\right| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| \to 0$$

as $n \to \infty$, but $\{a_n\}$ is not convergent because it is not bounded.

4. *Let $f : (0,1) \to \mathbf{R}$ be a function. Define: f is uniformly continuous on $(0,1)$. Using just the definition, show that f is uniformly continuous on $(0,1)$ where* $f(x) = \dfrac{x^2}{3-x}$.

$f : (0,1) \to \mathbf{R}$ is called *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \qquad \text{whenever } x, y \in (0,1) \text{ and } |x-y| < \delta.$$

Choose $\varepsilon > 0$. Let $\delta = \dfrac{4}{7}\varepsilon$. For any $x, y \in (0,1)$ such that $|x-y| < \delta$ we have $|x| \le 1$, $|y| \le 1$, $2 \le 3-x$ and $2 \le 3-y$ so that

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{x^2}{3-x} - \frac{y^2}{3-y}\right| \\
&= \frac{|(3-y)x^2 - (3-x)y^2|}{(3-x)(3-y)} \\
&= \frac{|3(x^2-y^2) - x^2y + xy^2|}{(3-x)(3-y)} \\
&= \frac{|3(x+y)(x-y) - xy(x-y)|}{(3-x)(3-y)} \\
&= \frac{|3x+3y-xy|\,|x-y|}{(3-x)(3-y)} \\
&\le \frac{(3|x| + 3|y| + |x|\,|y|)}{(3-x)(3-y)}|x-y| \\
&\le \frac{(3\cdot 1 + 3\cdot 1 + 1\cdot 1)}{2\cdot 2}|x-y| = \frac{7}{4}|x-y| < \frac{7}{4}\delta = \varepsilon. \quad \square
\end{aligned}$$

5. *Let $\{s_n\}$ be a real sequence. State the definition: $\{s_n\}$ is a* Cauchy Sequence. *Using just the definition, show that $\{s_n\}$ is a Cauchy Sequence, where the sequence of partial sums is defined for $n \in \mathbf{N}$ by*

$$s_n = \sum_{k=0}^{n} \frac{(-2)^k}{k!}$$

$\{s_n\}$ is a *Cauchy Sequence* if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|s_m - s_\ell| < \varepsilon \qquad \text{whenver } m, \ell > N.$$

We observe that for $k \ge 2$ that we have

$$\frac{|-2|^k}{k!} = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{4}\cdots\frac{2}{k} \le \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdots\frac{2}{3} = \frac{2\cdot 2}{1\cdot 2}\left(\frac{2}{3}\right)^{k-2} = \frac{9}{2}\left(\frac{2}{3}\right)^{k}.$$

To prove that the partial sums are a Cauchy Sequence, choose $\varepsilon > 0$. Let

$$N = \max\left\{2, \frac{\log\left(\frac{2\varepsilon}{27}\right)}{\log\left(\frac{2}{3}\right)} - 1\right\}.$$

For any $m, \ell \in \mathbf{N}$ such that $m, \ell > N$ we have either $m = \ell$ in which case $|s_m - s_\ell| = 0 < \varepsilon$. Or we have $m \ne \ell$. By swapping the names if necessary, we may assume that $m > \ell$. Then since $N \ge 2$ by using the triangle inequality and the observation

$$\begin{aligned}
|s_m - s_\ell| &= \left|\sum_{k=0}^{m}\frac{(-2)^k}{k!} - \sum_{k=0}^{\ell}\frac{(-2)^k}{k!}\right| \\
&= \left|\sum_{k=\ell+1}^{m}\frac{(-2)^k}{k!}\right| \\
&\le \sum_{k=\ell+1}^{m}\frac{|-2|^k}{k!} \\
&\le \sum_{k=\ell+1}^{m}\frac{9}{2}\left(\frac{2}{3}\right)^k \\
&= \frac{9}{2}\left(\sum_{k=0}^{m}\left(\frac{2}{3}\right)^k - \sum_{k=0}^{\ell}\left(\frac{2}{3}\right)^k\right) \\
&= \frac{9}{2}\left(\frac{1-\left(\frac{2}{3}\right)^{m+1}}{1-\frac{2}{3}} - \frac{1-\left(\frac{2}{3}\right)^{\ell+1}}{1-\frac{2}{3}}\right) \\
&= \frac{9}{2}\left(\frac{\left(\frac{2}{3}\right)^{\ell+1} - \left(\frac{2}{3}\right)^{m+1}}{1-\frac{2}{3}}\right) \\
&\le \frac{27}{2}\left(\frac{2}{3}\right)^{\ell+1} < \frac{27}{2}\left(\frac{2}{3}\right)^{N+1} \le \varepsilon. \quad \square
\end{aligned}$$