Math 3210 § 3. **Second Midterm Exam** Name: Solutions
Treibergs October 8, 2014

1. *Let $\{a_n\}$ be a real sequence and $L \in \mathbf{R}$. State the definition: $a_n \to L$ as $n \to \infty$. Find the limit. Using just the definition, prove that your answer is correct.* $L = \lim_{n\to\infty} \dfrac{3n + \sin n}{2n + \sin n}$.

   We say that $a_n \to L$ as $n \to \infty$ if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that $|a_n - L| < \varepsilon$ whenever $n > N$.

   We show that the limit is $L = \frac{3}{2}$. Choose $\varepsilon > 0$. Let $N = \frac{1}{4\varepsilon} + \frac{1}{2}$. For every $n \in \mathbf{N}$ such that $n > N$, since $n > N > \frac{1}{2}$ we have $2n > 1$ but $|\sin n| \le 1$ implies we also have $2n + \sin n \ge 2n - 1 > 0$. Hence

$$\begin{aligned}\left|\frac{3n + \sin n}{2n + \sin n} - \frac{3}{2}\right| &= \frac{|2(3n + \sin n) - 3(2n + \sin n)|}{2|2n + \sin n|}\\ &= \frac{|-\sin n|}{2(2n + \sin n)}\\ &\le \frac{1}{2(2n-1)}\\ &= \frac{1}{4n-2}\\ &< \frac{1}{4N-2} = \varepsilon. \quad \square\end{aligned}$$

2. *Let $E \subset \mathbf{R}$ be a nonempty subset and $f : E \to \mathbf{R}$ be a function. Define: $S = \inf_{x\in E} f(x)$. Find $\inf_{x\in E} f(x)$ and prove your answer, where $E = (0, \infty)$ and $f(x) = \dfrac{1+x}{x}$.*

   Definition of *infimum:* if f is not bounded below on E then $\inf_{x\in E} f(x) = -\infty$. Otherwise, the infimum is the greatest lower bound. First, S is a lower bound: $(\forall x \in E)(f(x) \ge S)$ and second, S is the greatest of lower bound: $(\forall b > S)(\exists x \in E)(f(x) < b)$.

   For this E and f we have $S = 1$. If $x \in E$ then $x > 0$ and $f(x) = \dfrac{1+x}{x} = 1 + \dfrac{1}{x} > 1 + 0$, so $S = 1$ is a lower bound. Choose $b > 1$ then for $x = \dfrac{2}{b-1} > 0$ so $x \in E$ we have

$$f(x) = \frac{1+x}{x} = \frac{1 + \frac{2}{b-1}}{\frac{2}{b-1}} = \frac{b-1}{2} + 1 = \frac{b+1}{2} < \frac{b+b}{2} = b. \quad \square$$

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement: *Let $\{a_n\}$ and $\{b_n\}$ be real sequences such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Assume $a_n < b_n$ for all n. Then $a < b$.*

   False. Let $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$. Then $a_n < b_n$ for all n but

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} b_n.$$

(b) STATEMENT: $|x| \geq |y| - |x-y|$ *for all* $x, y \in \mathbf{R}$.
TRUE. By the triangle inequality we have

$$|y| = |x + (y - x)| \leq |x| + |y - x|.$$

(c) STATEMENT: *Let* $\{x_n\}$ *be a real sequence. Suppose for every* $L \in \mathbf{R}$ *there is an* $n \in \mathbf{N}$ *such that* $x_n > L$. *Then* $\lim_{n\to\infty} x_n = \infty$.
FALSE. The given condition is for unboundedness above. The definition of $\lim_{n\to\infty} x_n = \infty$ is: $(\forall L \in \mathbf{R})(\exists N \in \mathbf{R})(\forall n \in \mathbf{N})(n > N \implies x_n > L)$.
Thus a counterexample is given by

$$x_n = \begin{cases} n, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

The condition holds: for every $L \in \mathbf{R}$ there is $m \in \mathbf{N}$ such that $m > L$ by the Archimidean Property, so $n = 2m > L$ and $x_n = n > L$ since n is even, but the limit is not infinite. For $L > 0$ it is not true that there is an $N \in \mathbf{R}$ such that $x_n > L$ for all $n > N$ because, no matter what N is, there are odd numbers n with $n > N$, such that $x_n = 0$ which is not greater than L.

4. *Let* $\{a_n\}$ *be a real sequence and* $a \in \mathbf{R}$. *Suppose* $a_n \to a$ *as* $n \to \infty$. *Using just the definition of limit (and not the Main Limit Theorem), show that cube sequence converges*

$$(a_n)^3 \to a^3 \qquad \text{as } n \to \infty.$$

Choose $\varepsilon > 0$. Since $a_n \to a$, for $\varepsilon' = 1$, there is an $N_1 \in \mathbf{R}$ such that

$$|a_n - a| < \varepsilon' = 1, \qquad \text{whenever } n > N_1.$$

For such n we have by the triangle inequality

$$|a_n| = |a + (a_n - a)| \leq |a| + |a_n - a| < |a| + \varepsilon' = |a| + 1.$$

Also since $a_n \to a$, for $\varepsilon'' = \dfrac{\varepsilon}{3(|a|+1)^2}$ there is $N_2 \in \mathbf{R}$ such that

$$|a_n - a| < \varepsilon'' = \frac{\varepsilon}{3(|a|+1)^2} \qquad \text{whenever } n > N_2.$$

Let $N = \max\{N_1, N_2\}$. For any $n \in \mathbf{N}$ satisfying $n > N$, since $n > N_1$ we have $|a_n| \leq |a|+1$ and since also $n > N_2$,

$$\begin{aligned} |a_n^3 - a^3| &= |(a_n - a)(a_n^2 + a_n a + a^2)| \\ &\leq |a_n - a| \left(|a_n|^2 + |a_n|\,|a| + |a|^2\right) \\ &\leq |a_n - a| \cdot \left((|a|+1)^2 + (|a|+1)|a| + |a|^2\right) \\ &\leq |a_n - a| \cdot 3(|a|+1)^2 \\ &< \frac{\varepsilon}{3(|a|+1)^2} \cdot 3(|a|+1)^2 = \varepsilon. \quad \square \end{aligned}$$

5. *Define a sequence recursively by* $a_1 = 1$ *and* $a_{n+1} = 1 - \dfrac{1}{2 + a_n}$. *Prove that the sequence* $\{a_n\}$ *converges. What is* $\lim_{n\to\infty} a_n$*? Why?*

Computing the first several terms we find $a_1 = 1$, $a_2 = \frac{2}{3}$. $a_3 = \frac{5}{8}$. $a_4 = \frac{13}{21}$, which suggests a_n is decreasing. We shall show that $\{a_n\}$ is decreasing and bounded below. Hence, by the Monotone Convergence Theorem, there is $a \in \mathbf{R}$ such that $a_n \to a$ as $n \to \infty$.

First we show that $a_n > 0$ for all $n \in \mathbf{N}$ so that $\{a_n\}$ is bounded below by zero. Argue by induction. Base case: the first term is defined $a_1 = 1$ so it is greater than zero. Induction case: assume for some $n \in \mathbf{N}$ that $a_n > 0$. Then

$$a_{n+1} = 1 - \frac{1}{2 + a_n} = \frac{1 + a_n}{2 + a_n} = \frac{(+)}{(+)} > 0$$

since both numerator and denominator are positive by the induction hypothesis. This completes the argument that $a_n > 0$ for all $n \in \mathbf{N}$.

Next we show that a_n is decreasing by induction. Base case: we have

$$a_2 = 1 - \frac{1}{2 + a_1} = 1 - \frac{1}{2 + 1} = \frac{2}{3} < 1 = a_1$$

so that $a_2 - a_1 < 0$. Induction case: assume that for any $n \in \mathbf{N}$ we have $a_{n+1} - a_n < 0$. Then

$$\begin{aligned}
a_{n+2} - a_{n+1} &= \left(1 - \frac{1}{2 + a_{n+1}}\right) - \left(1 - \frac{1}{2 + a_n}\right) \\
&= \frac{-(2 + a_n) + (2 + a_{n+1})}{(2 + a_{n+1})(2 + a_n)} \\
&= \frac{a_{n+1} - a_n}{(2 + a_n)(2 + a_{n+1})} \\
&= \frac{(-)}{(+)(+)} < 0,
\end{aligned}$$

where we have used the induction hypothesis on the numerator and the positivity of a_n in the denominator. This completes the argument that a_n is decreasing.

Finally, we compute a. By the Subsequences Theorem we see that $a_{n+1} \to a$ as $n \to \infty$. Taking limits of both sides of the recursion equation yields by the Main Limit Theorem,

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(1 - \frac{1}{2 + a_n}\right) = 1 - \frac{1}{2 + a}.$$

Solve for a by cross multiplying $a(2 + a) = (2 + a) - 1$ so $a^2 + a - 1 = 0$. By the quadratic formula

$$a = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $a_n > 0$ for all n we have $a = \lim_{n\to\infty} a_n \geq 0$ so only the positive root gives the limit $a = -\frac{1}{2} + \frac{\sqrt{5}}{2} = 0.61803$. $\square$