Math 3210 § 2. **First Midterm Exam** Name: Solutions
Treibergs January 29, 2014

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1. *Let $\{x_0, x_1, x_2, \ldots\}$ be a sequence defined recursively by $x_0 = 1$, $x_1 = 2$, and for $n \in \mathbb{N}$, $x_{n+1} = 3x_n - 2x_{n-1}$. Prove that $x_n = 2^n$ for every integer $n \geq 0$.*

For $n \in \mathbb{N}$, let the statement

$$\mathcal{P}_n = \text{“}x_n = 2^n \text{ and } x_{n-1} = 2^{n-1}.\text{”}$$

We prove it for all n by induction.

Base case $n = 1$: We are given $x_1 = 2 = 2^1$ and $x_0 = 1 = 2^0$, so $\mathcal{P}_1$ is true.

Induction Case: Assume that for some $n \in \mathbb{N}$, $\mathcal{P}_n$ is true to show that $\mathcal{P}_{n+1}$ is also true. Thus we are assuming $x_n = 2^n$ and $x_{n-1} = 2^{n-1}$ which says $x_{(n+1)-1} = 2^{(n+1)-1}$ so that the second equation of $\mathcal{P}_{n+1}$ holds. Using the recursion and the induction hypothesis

$$x_{n+1} = 3x_n - 2x_{n-1} = 3 \cdot 2^n - 2 \cdot 2^{n-1} = 3 \cdot 2^n - 2^n = 2 \cdot 2^n = 2^{n+1}$$

so that the first equation of $\mathcal{P}_{n+1}$ is also true. thus the induction case is done.

Hence $\mathcal{P}_n$ holds for all $n \in \mathbb{N}$, so $x_n = 2^n$ for all $n \in \mathbb{N}$.

2. *Using only the axioms of a commutative ring, show that for every $a, b \in R$, if $a = a + b$ then $b = 0$. Justify every step of your argument using just the axioms listed here. Use ONLY the axioms listed and DO NOT SKIP STEPS.*

Recall the axioms of a commutative ring $(R, +, \times)$. For any $x, y, z \in R$,

- A1. (Commutativity of Addition.) $x + y = y + x$.
- A2. (Associativity of Addition.) $x + (y + z) = (x + y) + z$.
- A3. (Additive Identity.) $(\exists\, 0 \in R)\,(\forall\, t \in R)\ 0 + t = t$.
- A4. (Additive Inverse) $(\exists\, {-x} \in R)\ x + (-x) = 0$.
- M1. (Commutativity of Multiplication.) $xy = yx$.
- M2. (Associativity of Multiplication.) $x(yz) = (xy)z$.
- M3. (Multiplicative Identity.) $(\exists\, 1 \in R)\ 1 \neq 0$ and $(\forall\, t \in R)\ 1t = t$.
- D. (Distributivity) $x(y + z) = xy + xz$.

$a = a + b$	Assumption.
$a + (-a) = (a + b) + (-a)$	By A4 there is $-a$. Add to both sides.
$a + (-a) = (b + a) + (-a)$	By A1.
$a + (-a) = b + (a + (-a))$	By A2.
$0 = b + 0$	By A4.
$0 = 0 + b$	By A1.
$0 = b$	By A3.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

a. **Statement:** *If $f : A \to B$ and $A = f^{-1}(B)$ then f is onto.*

FALSE. Let $A = \{1\}$, $B = \{2, 3\}$, $f(1) = 2$. Then $f^{-1}(B) = \{1\} = A$ but f is not onto because $3 \in B$ is not the value, $3 \neq f(x)$, of any $x \in A$.

b. **Statement:** *Let $f : X \to Y$ and $A, B \subset X$ be subsets. If $f(A) \cap f(B) \neq \emptyset$ then $A \cap B \neq \emptyset$.*

FALSE. Let $X = \{1, 2\}$, $Y = \{3\}$, $f(1) = f(2) = 3$, $A = \{1\}$ and $B = \{2\}$. Then $f(A) \cap f(B) = \{3\} \cap \{3\} = \{3\} \neq \emptyset$ but $A \cap B = \emptyset$.

c. **Statement:** *Suppose $A, B \subset X$. Then $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$.*

TRUE. This is deMorgan's formula. Using the deMorgan's formula from logic,

$$\begin{aligned}
x \in X\backslash(A \cup B) &\iff x \in X \text{ and } x \notin (A \cup B)\\
&\iff x \in X \text{ and } \sim \big(x \in (A \cup B)\big)\\
&\iff x \in X \text{ and } \sim \big(x \in A \text{ or } x \in B\big)\\
&\iff x \in X \text{ and } \big(\sim (x \in A) \text{ and } \sim (x \in B)\big)\\
&\iff x \in X \text{ and } \big(x \notin A \text{ and } x \notin B\big)\\
&\iff (x \in X \text{ and } x \notin A) \text{ and } (x \in X \text{ and } x \notin B)\\
&\iff (x \in X\backslash A) \text{ and } (x \in X\backslash B)\\
&\iff x \in (X\backslash A) \cap (X\backslash B)\,.
\end{aligned}$$

4. *Let $(F, +, \times)$ be a field with order relation "$\leq$." How is $x < y$ defined? Using properties of a field and the order axioms, show that if $x, y, z \in F$ satisfy $x < y$ and $0 < z$ then $xz < yz$.*

Recall that "$\leq$" is a relation that satisfies the following axioms: for all $x, y, z \in F$,

O1. Either $x \leq y$ or $y \leq x$.

O2. If $x \leq y$ and $y \leq x$ then $x = y$.

O3. If $x \leq y$ and $y \leq z$ then $x \leq z$.

O4. If $x \leq y$ then $x + z \leq y + z$.

O5. If $x \leq y$ and $0 \leq z$ then $xz \leq yz$.

$x < y$ means $x \leq y$ and $x \neq y$.

We assume $x < y$ and $0 < z$. By definition of "$<$," this means $x \leq y$ and $x \neq y$ and $0 \leq z$ and $0 \neq z$. Since $x \leq y$ and $0 \leq z$ we have that $xz \leq yz$ by **O5**.

Also we have $0 \neq z$ and we wish to show $x \neq y$ implies $xz \neq yz$. By contraposition, this is equivalent to showing $xz = yz$ implies $x = y$. Since $z \neq 0$, by the multiplicative inverse in the field, there is z^{-1} such that $z^{-1}z = 1$.

$xz = yz$	Assumption.
$z^{-1}(xz) = z^{-1}(yz)$	By M4 there is z^{-1}. Multiply both sides.
$z^{-1}(zx) = z^{-1}(zy)$	By M1.
$(z^{-1}z)x = (z^{-1}z)y$	By M2.
$1x = 1y$	By M4.
$x = y$	By M3.

Hence we have shown that $xz \leq yz$ and $xz \neq yz$. It follows that $xz < yz$ as to be shown.

5. *For* ϵ*,* δ *real, let* E *the given subset of the real numbers. Determine* E*. Prove that your set equals the given* E*.*

$$E = \big\{ x \in \mathbf{R} : \ \big[(\forall \epsilon > 0) \ (x < \epsilon) \big] \text{ and } \big[(\exists \delta > 0) \ (-\delta < x) \big] \big\}$$

We can see what E is by replacing it with unions and intersections of intervals.

$$E = \left(\bigcap_{\epsilon > 0} (-\infty, \epsilon) \right) \cap \left(\bigcup_{\delta > 0} (-\delta, \infty) \right) = (-\infty, 0] \cap \mathbf{R} = (-\infty, 0].$$

To prove $E = (-\infty, 0]$ we argue "$\subset$" and "$\supset$."

To show $(-\infty, 0] \subset E$, we choose $x \in (-\infty, 0]$. Hence $x \le 0$. It follows that $x < \epsilon$ for every $\epsilon > 0$. Also, let $\delta = -x + 1$. Since $x \le 0$, $\delta > 0$. Also, $-\delta = x - 1 < x$. Thus we have shown there is a $\delta > 0$ so that $-\delta < x$. Both conditions defining E hold so $x \in E$.

To show $E \subset (-\infty, 0]$ or $x \in E$ implies $x \in (-\infty, 0]$ we argue the contrapositive: if $x \notin (-\infty, 0]$ then $x \notin E$. But an arbitrary $x \notin (-\infty, 0]$ means that $x > 0$. But then let $\epsilon = x > 0$. Thus there is $\epsilon > 0$ such that $\sim (x < \epsilon)$. In other words $(\forall \epsilon > 0) \ (x < \epsilon)$ is false. Thus one of the conditions to be in E is violated. However, since both must hold for a point to be in E, it follows that $x \notin E$, as to be proved.

Thus we have shown both containments, so $E = (-\infty, 0]$.