Math 3210 § 3. **Final Exam** Name: Solutions
Treibergs December 16, 2009

1. *Let $f : \mathbb{R} \to \mathbb{R}$. Define: f is* differentiable *at a. Determine whether the given function is differentiable at 0. Justify your answer.*

$$f(x) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+x^4}}, & \text{if } x \neq 0; \\ \\ 0, & \text{otherwise.} \end{cases}$$

Definition: f is differentiable at a if the following limit exists: $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$.

For the given function, the limit does not exist at 0. To see this we show for two sequences tending to zero, the difference quotient has different limits. Taking $x_n = \dfrac{1}{n}$,

$$\frac{f(x_n)-f(0)}{x_n - 0} = \frac{x_n^2}{x_n\sqrt{x_n^2+x_n^4}} = \frac{x_n}{\sqrt{x_n^2(1+x_n^2)}} = \frac{x_n}{|x_n|\sqrt{1+x_n^2}} = \frac{\frac{1}{n}}{\left|\frac{1}{n}\right|\sqrt{1+\frac{1}{n}^2}} = \frac{1}{\sqrt{1+\frac{1}{n}^2}} \to 1$$

as $n \to \infty$. On the other hand, for $y_n = -\dfrac{1}{n}$,

$$\frac{f(y_n)-f(0)}{y_n - 0} = \frac{y_n^2}{y_n\sqrt{y_n^2+y_n^4}} = \frac{-\frac{1}{n}}{\left|-\frac{1}{n}\right|\sqrt{1+\frac{1}{n}^2}} = \frac{-1}{\sqrt{1+\frac{1}{n}^2}} \to -1$$

as $n \to \infty$. Since the left and right approaches have inconsistent limits, there is no limit so the function is not differentiable at 0. □

2. *Suppose $f : [0, 2\pi] \to \mathbb{R}$ is a continuous function and that $f(q) = 0$ for every rational number $q \in [0, 2\pi] \cap \mathbb{Q}$. Show that $f(x) = 0$ for all $x \in [0, 2\pi]$.*

Fix an arbitrary $x \in [0, 2\pi]$. We show for every $\varepsilon > 0$, we have $|f(x)| < \varepsilon$, thus $f(x) = 0$.

Choose $\varepsilon > 0$. By continuity of f at x, there is a $\delta > 0$ so that

$$|f(x) - f(y)| < \varepsilon \qquad \text{whenever } y \in [0, 2\pi] \text{ and } |x - y| < \delta.$$

By the density of rationals, there is a $q \in \mathbb{Q} \cap [0, 2\pi]$ so that $|x - q| < \delta$. Thus for this q, since $f(q) = 0$,

$$|f(x)| = |f(x) - 0| = |f(x) - f(q)| < \varepsilon.$$

Since ε was arbitrary, $f(x) = 0$. □

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a.) Statement: *Let $(\mathcal{F}, +, , 0, 1)$ be a field. If $x, y \in \mathcal{F}$ such that $x \neq 0$ satisfy $x \cdot y = x$ then $y = 1$.*

TRUE. Since $x \neq 0$ it has an inverse x^{-1}. Premultiplying the equation, $x^{-1}(xy) = x^{-1}x$, so by associativity $(x^{-1}x)y = x^{-1}x$, by multiplicative inverse $1 \cdot y = 1$ and by multiplicative identity, $y = 1$.

(b.) Statement: *The sequence $\left\{\dfrac{n-1}{n}\right\}$ is a Cauchy Sequence.*

TRUE. It converges $\dfrac{n-1}{n} \to 1$ as $n \to \infty$, thus is a Cauchy Sequence.

(c.) Statement: *If* $f_n, g : \mathbb{R} \to \mathbb{R}$ *are functions such that* $|f_n(x) - g(x)| < \dfrac{1}{2^n}$ *for all* $x \in \mathbb{R}$ *and for all* $n \in \mathbb{N}$*. Then* $f_n \to g$ *uniformly in* $\mathbb{R}$*.*

TRUE. Choose $\varepsilon > 0$. Let $R \in \mathbb{R}$ be such that $\frac{1}{2^R} < \varepsilon$. then for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$ such that $n > R$ we have

$$|f_n(x) - g(x)| < \frac{1}{2^n} < \frac{1}{2^R} < \varepsilon.$$

But this is the definition of f_n converging uniformly to g on $\mathbb{R}$.

4. *Let* $E = \left\{ \int_0^x f(x) \sin x \, dx \,\middle|\, f : [0, \pi] \to \mathbb{R} \text{ is continuous and } f(x) > 0 \text{ for all } x \in [0, \pi] \right\}$*. Show that* E *is nonempty and bounded below. What is the greatest lower bound of* E*? Does the set* E *have a minimum? Justify your answers.*

Since $f(x)$ is continuous and $\sin x$ is continuous, both are integrable, hence their product $f(x) \sin x$ is integrable and its integral has a real value in E, showing $E \neq \emptyset$. We show E is bounded below by zero, namely for all $z \in E$ we have $z \geq 0$. Observe that $\sin x \geq 0$ since $0 \leq x \leq \pi$. Also $f(x) > 0$ for such x. Hence $f(x) \sin x \geq 0$. Integrating

$$\int_0^\pi f(x) \sin x \, dx \geq 0.$$

Since all numbers in E are of this form, 0 is a lower bound for E.

Second we show that zero is the greatest lower bound. We show for every $\varepsilon > 0$ there is a $z \in E$ such that $z < 0 + \varepsilon$. Hence positive numbers are not lower bounds. Choose $\varepsilon > 0$. Then $f(x) = \dfrac{\varepsilon}{2\pi}$ is a continuous, positive function. Because $\sin x \leq 1$ for $x \in [0, \pi]$ we have $f(x) \sin x \leq \dfrac{\varepsilon}{2\pi}$ for $0 \leq x \leq \pi$. Then the element $z \in E$ given by

$$z = \int_0^\pi f(x) \sin x \, dx \leq \int_0^\pi \frac{\varepsilon}{2\pi} \, dx = \frac{\varepsilon}{2} < \varepsilon.$$

Thus ε is not a lower bound so 0 must be the greatest lower bound.

Third, the set E does not have a minimum since $z > 0$ for all $z \in E$. To see this, choose $z \in E$. Thus $z = \displaystyle\int_0^x f(x) \sin x \, dx$ for some continuous, positive f. Since f is continuous on $[0, \pi]$ it takes its minimum: there is a $c \in [0, \pi]$ such that $f(c) = \inf_{[0,\pi]} f$. But since f is positive, $f(c) > 0$. But $\sin x \geq 0$ implies $f(x) \sin x \geq f(c) \sin x$ for $0 \leq x \leq \pi$, it follows that $z > 0$ because

$$z = \int_0^\pi f(x) \sin x \, dx \geq f(c) \int_0^\pi \sin x \, dx = 2f(c) > 0.$$

5. *Prove that if* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable on* $\mathbb{R}$ *and* $f'(x)$ *is bounded on* $\mathbb{R}$*, then* f *is uniformly continuous on* $\mathbb{R}$*.*

We show that f is uniformly continuous, namely, for all $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \mathbb{R}$ and $|x - y| < \delta$. Choose $\varepsilon > 0$. Because f' is bounded, there is an $M \in \mathbb{R}$ so that $|f'(c)| \leq M$ for all $x \in \mathbb{R}$. Let $\delta = \dfrac{\varepsilon}{1 + M}$. Suppose that $x, y \in \mathbb{R}$ such that $|x - y| < \delta$. If $x = y$ then $|f(x) - f(y)| = 0 < \varepsilon$ and we are done. If $x \neq y$, for convenience we may assume that $x < y$ by swapping roles if necessary. Now, as it is differentiable, f is continuous on $\mathbb{R}$. Hence it is continuous on $[x, y]$ and differentiable on (x, y) because these are subintervals of $\mathbb{R}$. Hence we may apply the Mean Value Theorem: there is a $c \in (x, y)$ so that $f(y) - f(x) = f'(c)(y - x)$. It follows that

$$|f(y) - f(x)| = |f'(c)||y - x| \leq M|y - x| < M \cdot \frac{\varepsilon}{1 + M} < \varepsilon. \quad \square$$

5. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a.) Statement: *If $f : [0, \infty) \to \mathbb{R}$ is continuous, positive and $f(x) \to 0$ as $x \to \infty$ then the improper integral $\displaystyle\int_0^\infty f(x)\,dx$ converges.*

FALSE. The function $f(x) = \dfrac{1}{1+x}$ is continuous on $[0, \infty)$ and tends to zero as $x \to \infty$. But its improper integral does not converge:

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx = \lim_{R\to\infty} \int_0^R \frac{dx}{1+x} = \lim_{0\to R} \ln(1+R) = \infty.$$

(b.) Statement: *If $\displaystyle\sum_{k=1}^\infty a_k$ and $\displaystyle\sum_{k=1}^\infty b_k$ are convergent series then $\displaystyle\sum_{k=1}^\infty [a_k + b_k]$ is a convergent series.*

TRUE. Because the finite sum is additive, we may deduce the result from the sum theorem for limits:

$$\begin{aligned}
\sum_{k=1}^\infty [a_k + b_k] &= \lim_{n\to\infty} \sum_{k=1}^N [a_k + b_k] \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^N a_k + \sum_{k=1}^N b_k \right) \\
&= \left(\lim_{n\to\infty} \sum_{k=1}^N a_k \right) + \left(\lim_{n\to\infty} \sum_{k=1}^N b_k \right) \\
&= \left(\sum_{k=1}^\infty a_k \right) + \left(\sum_{k=1}^\infty b_k \right).
\end{aligned}$$

(c.) Statement: *If $f : [0, 1] \to \mathbb{R}$ is nonnegative and bounded, then it is integrable on $[0, 1]$.*

FALSE. The Dirichlet Function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \notin \mathbb{Q}; \end{cases}$$

satisfies $0 \le f(x) \le 1$ so is nonnegative and bounded. It is also not integrable. Any lower sum is dead zero so $\underline{\int_0^1} f\,dx = 0$ and any upper sum is one so $\overline{\int_0^1} f\,dx = 1$ which are not equal.

7. *Let $a < b$ and $f : [a, b] \to \mathbb{R}$. Show that if for all $\varepsilon > 0$ there are integrable functions $g, h : [a, b] \to \mathbb{R}$ such that $g(x) \le f(x) \le h(x)$ for all $x \in [a, b]$ and $\displaystyle\int_a^b h(x) - g(x)\,dx < \varepsilon$ then f is integrable on $[a, b]$.*

We use the theorem characterizing integrability: the bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that the upper and lower sums satisfy $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$.

If $\mathcal{P}$ is a partition and I_k is one of the subintervals of the partition, denote by $M_k(f) = \sup\{f(x) : x \in I_k\}$ and by $m_k(f) = \inf\{f(x) : x \in I_k\}$.

Choose $\varepsilon > 0$. Let g and h be the given integrable functions such that $g(x) \le f(x) \le h(x)$ and $\int_a^b h(x) - g(x)\,dx < \frac{\varepsilon}{3}$. Choose a partition $\mathcal{P}'$ such that $U(g,\mathcal{P}') - L(g,\mathcal{P}') < \frac{\varepsilon}{3}$. Choose a partition $\mathcal{P}''$ such that $U(h,\mathcal{P}'') - L(h,\mathcal{P}'') < \frac{\varepsilon}{3}$. Let $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$ be the common refinement. Since refining increases lower sums and degreases upper sums, we have $U(g,\mathcal{P}) - L(g,\mathcal{P}) < \frac{\varepsilon}{3}$ and $U(h,\mathcal{P}) - L(h,\mathcal{P}) < \frac{\varepsilon}{3}$. Also, the integral falls between the lower sum and upper sum, so we have

$$\left| L(g,\mathcal{P}) - \int_a^b g\,dx \right| < \frac{\varepsilon}{3}, \qquad \text{and} \qquad \left| U(h,\mathcal{P}) - \int_a^b h\,dx \right| < \frac{\varepsilon}{3}.$$

Now let's estimate the lower and upper sum for f. Because $g \le f$ we have $m_k(g) \le m_k(f)$, so by summing,

$$L(g,\mathcal{P}) = \sum_{k=1}^{n} m_k(g)(x_k - x_{k-1}) \le \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) = L(f,\mathcal{P}).$$

Similarly, because $f \le h$ we have $M_k(f) \le M_k(h)$, so by summing, $U(f,\mathcal{P}) \le U(h,\mathcal{P})$. Now, assemble the inequalities. For the partition $\mathcal{P}$ we have

$$\begin{aligned} U(f,\mathcal{P}) - L(f,\mathcal{P}) &\le U(h,\mathcal{P}) - L(g,\mathcal{P}) \\ &= \int_a^b h + \left(U(h,\mathcal{P}) - \int_a^b h \right) - \int_a^b g - \left(L(g,\mathcal{P}) - \int_a^b g \right) \\ &\le \int_a^b (\,h - g\,) + \left| U(h,\mathcal{P}) - \int_a^b h \right| + \left| L(g,\mathcal{P}) - \int_a^b g \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \square \end{aligned}$$

8. *Determine whether the following series are absolutely convergent, conditionally convergent or divergent. In each case you must justify your answer.*

(a.) $S = \sum_{k=1}^{\infty} (-1)^k \frac{\log k}{k}$.

CONVERGENT. Recall the definitions. If $\sum_{k=1}^{\infty} a_k$ is a series then the series is *absolutely convergent* if the series of absolute values converges, namely the sequence of absolute partial sums has a finite limit: $\lim_{n\to\infty} \sum_{k=1}^{n} |a_k|$ converges. The series is *convergent* if the sequence of partial sums itself has a finite limit: $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ converges. Absolute convergence implies convergence, proved *e.g.,* using the Cauchy Criterion. The series is *divergent* if it is not convergent.

The $(-1)^k$ make the terms alternate signs. The magnitude of the summand $\frac{\log k}{k}$ decreases and tends to zero. To see it, let $f(x) = \frac{\log x}{x}$. Then $f'(x) = \frac{1 - \log x}{x^2} < 0$ if $x > e$. Hence $f(k)$ is strictly decreasing and positive for $k \ge 3$. By l'Hopital's Rule, $\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$ so $f(k) \to 0$ as $k \to \infty$. Thus the conditions for the alternating series test hold and S converges.

However, $f(k) \ge \frac{1}{k}$ for $k \ge 3$ so that $\sum_{k=3}^{n} f(k) \ge \sum_{k=3}^{n} \frac{1}{k} \to \infty$ as $n \to \infty$ because the harmonic series diverges to infinity.

(b.) $S = \sum_{k=1}^{\infty} \frac{(-1)^k \log k}{\log(k^2+k+1)}$.

DIVERGENT. A necessary condition for the convergence of an infinite sum is that the terms tend to zero. However, letting $f(x) = \frac{\log x}{\log(x^2+x+1)}$, by l'Hopital's Rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2x+1}{x^2+x+1}} = \lim_{x\to\infty} \frac{x^2+x+1}{(2x+1)x} = \frac{1}{2}.$$

Thus the terms $(-1)^k f(k)$ do not tend to zero and the series is divergent.

(c.) $S = \sum_{k=1}^{\infty} (-1)^k \frac{\log k}{k^2}$.

ABSOLUTELY CONVERVENT. Let $f(x) = \frac{\log x}{x^2}$. The absloute sum is convergent by the integral test. Since $f'(x) = \frac{1 - 2\log x}{x^3} < 0$ for $x \ge 2$, we can compare the partial sum with the integral. By substituting $u = \log x$,

$$\sum_{k=3}^{n} f(k) \le \int_1^n \frac{\log x\, dx}{x^2} = \int_0^{\log n} ue^{-u}\, du = 1 - \frac{1+\log n}{n} \le 1$$

for all $n \ge 3$. Since the absolute partial sums form a nondecreasing sequence, it is convergent because it is bounded above.

9. *Prove that if* $\sum_{k=1}^{\infty} a_k$ *is an absolutly convergent series and if* $\{b_k\}$ *is a bounded then* $\sum_{k=1}^{\infty} a_k b_k$ *is an absolutly convergent series.*

$\sum_{k=1}^{\infty} a_k$ is *absolutely convergent* if the series of absolute values converges, namely $\lim_{n\to\infty} \sum_{k=1}^{n} |a_k|$ converges to a finite limit.

Let's prove the Cauchy Criterion for convergence. Put $S_n = \sum_{j=1}^{n} |a_j b_j|$, $T_n = \sum_{j=1}^{n} |a_j|$. Because $\{b_j\}$ is bounded, there is an $M \in \mathbb{R}$ so that $|b_j| \le M$ for all $j \in \mathbb{N}$. Since the series is absolutely convergent, it is a Cauchy Sequence: for every $\varepsilon > 0$, there is an $R \in \mathbb{R}$ so that

$$|T_k - T_\ell| < \frac{\varepsilon}{1+M} \qquad \text{whenever } k, \ell \in \mathbb{N} \text{ such that } k, \ell > R.$$

Now suppose that $k, \ell \in \mathbb{N}$ such that $k, \ell > R$. If $k = \ell$ then $|S_k - S_\ell| = 0 < \varepsilon$ so we are done. If

$k \neq \ell$, we may swap roles if necessary to arrange that $k < \ell$. Thus

$$\begin{aligned}
|S_\ell - S_k| &= \left| \sum_{j=1}^{\ell} |a_j b_j| - \sum_{k=1}^{k} |a_j b_j| \right| \\
&= \sum_{j=k+1}^{\ell} |a_j|\,|b_j| \\
&\leq M \sum_{j=k+1}^{\ell} |a_j| \\
&= M \left| \sum_{j=1}^{\ell} |a_j| - \sum_{k=1}^{k} |a_j| \right| \\
&= M\,|T_\ell - T_k| < M \frac{\varepsilon}{1+M} < \varepsilon. \quad \square
\end{aligned}$$