Math 3210 § 2. **Third Midterm Exam Sample** Name: Sample
Treibergs November 6, 2009

(1.)*Show directly from the definition that* $\{x_n\}_{n\in\mathbf{N}}$ *is a Cauchy sequence, where*

$$x_n = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots + \frac{1}{n^n} = \sum_{j=1}^{n} \frac{1}{j^j},$$

Proof. To show that $\{x_n\}_{n\in\mathbf{N}}$ is Cauchy, which means for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ so that for every $k, \ell \ge N$, there holds $|x_k - x_\ell| < \varepsilon$.

Choose $\varepsilon > 0$. Let $N = 1/\varepsilon$. Suppose we choose $k, \ell \in \mathbf{N}$ so that $k, \ell \ge N$. If $k < \ell$ then swap the names of the numbers. Thus we may assume $k \ge \ell$. If $k = \ell$ then $|x_k - x_\ell| = 0 < \varepsilon$. If $k > \ell$ then

$$|x_k - x_\ell| = \left| \sum_{j=1}^{k} \frac{1}{j^j} - \sum_{j=1}^{\ell} \frac{1}{j^j} \right| = \sum_{j=\ell+1}^{k} \frac{1}{j^j} \le \sum_{j=\ell+1}^{k} \frac{1}{2^j} = \frac{1}{2^\ell} - \frac{1}{2^k} \le \frac{1}{2^\ell} < \frac{1}{\ell} \le \frac{1}{N} < \varepsilon.$$

where we have used $j^j \ge 2^j$ for $j \ge \ell + 1$ (≥ 2), $2^\ell > \ell$ (which is problem 1.2.6a) and the sum of a geometric series $\sum_{j=\ell+1}^{k} r^j = (r^{\ell+1} - r^{k+1})/(1-r)$.

(2.)*For each* $n \in \mathbf{N}$, *suppose that* $a_n \in \mathbf{R}$ *satisfies* $|a_n| \le n$. *Show that the sequence* $\{r_n\}_{n\in\mathbf{N}}$ *where* $r_n = a_n/n$ *has a convergent subsequence.*

Proof. We show that $\{r_n\}_{n\in\mathbf{N}}$ is a bounded sequence. Indeed, for all $n \in \mathbf{N}$, by the hypothesis $|a_n| \le n$,

$$|r_n| = \left| \frac{a_n}{n} \right| = \frac{|a_n|}{n} \le \frac{n}{n} = 1.$$

By the Bolzano-Weierstraß Theorem 2.26, the bounded sequence $\{r_n\}_{n\in\mathbf{N}}$ has a convergent subsequence. □

(3.)*Suppose that the real sequence* $\{x_n\}_{n\in\mathbf{N}}$ *is bounded and that the real sequence* $\{y_n\}_{n\in\mathbf{N}}$ *tends to infinity* $y_n \to \infty$ *as* $n \to \infty$. *Show*

$$\lim_{n\to\infty} (x_n + y_n) = \infty, \qquad [\text{i.e. } \quad x + \infty = \infty.]$$

Proof. We show that $z_n = x_n + y_n \to \infty$ as $n \to \infty$ which means for all $M \in \mathbf{R}$ there is an $N \in \mathbf{N}$ so that for every $k \in \mathbf{N}$ such that $k \ge N$ we have $z_k > M$.

As $\{x_n\}_{n\in\mathbf{N}}$ is a bounded sequence, there is a $C \in \mathbf{R}$ so that $|x_k| \le C$ for all $k \in \mathbf{N}$. Choose $M \in \mathbf{R}$. As $\{y_n\}_{n\in\mathbf{N}}$ diverges to infinity as $n \to \infty$, there is an $N \in \mathbf{N}$ so that for every $k \in \mathbf{N}$ such that $k \ge N$ we have $y_k > M + C$. We show that this N proves the claim for $\{z_n\}_{n\in\mathbf{N}}$. Thus if we choose $k \in \mathbf{N}$ such that $k \ge N$ then

$$z_k = y_k + x_k > (M + C) - |x_k| \ge (M + C) - C = M. \quad \square$$

(4.)*Suppose* $\{x_n\}_{n\in\mathbf{N}}$ *is a Cauchy Sequence such that some subsequence* $x_{n_j} \to L$ *as* $j \to \infty$. *Then the full sequence converges* $x_n \to L$ *as* $n \to \infty$.

Proof. We show that $x_n \to L$ as $n \to \infty$ which means, for all $\varepsilon > 0$ there is an $N \in \mathbf{N}$ so that for all $k \in \mathbf{N}$ such that $k \ge N$ we have $|x_k - L| < \varepsilon$.

Choose $\varepsilon > 0$. As $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy Sequence, there is a $K \in \mathbf{N}$ so that for all $k, \ell \in \mathbf{N}$ such that $k, \ell \ge K$ we have $|x_k - x_\ell| < \frac{1}{2}\varepsilon$. As the subsequence $x_{n_j} \to L$ as $j \to \infty$, there is a $J \in \mathbf{N}$ such that for every $j \in \mathbf{N}$ such that $j \ge J$ we have $|x_{n_j} - L| < \frac{1}{2}\varepsilon$. Now $N = \max\{K, n_J\}$ is the number that proves the convergence. Choose any $k \in \mathbf{N}$ such that $k \ge N$. Let $\ell = n_N$. We have $\ell = n_N \ge N \ge n_J \ge J$. Then, by the triangle inequality,

$$|x_k - L| = |(x_k - x_\ell) + (x_\ell - L)| \le |x_k - x_\ell| + |x_{n_N} - L| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \quad \square$$

(5.)*Show directly from the definition that* $\{x_n\}_{n\in\mathbf{N}}$ *is a Cauchy Sequence, where*

$$x_n = \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots + \frac{1}{(2n)!} = \sum_{j=1}^{n} \frac{1}{(2j)!}.$$

Proof. To show that $\{x_n\}_{n\in\mathbf{N}}$ is Cauchy, which means for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ so that for every $k, \ell \in \mathbf{N}$ such that $k, \ell \geq N$, there holds $|x_k - x_\ell| < \varepsilon$.

Choose $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. Choose $k, \ell \in \mathbf{N}$ so that $k, \ell \geq N$. If $k = \ell$ then $|x_k - x_\ell| = 0 < \varepsilon$. If $k \neq \ell$, by swapping roles if needed, we may assume $k > \ell$. Then

$$\begin{aligned}
|x_k - x_\ell| &= \left| \sum_{j=1}^{k} \frac{1}{(2j)!} - \sum_{j=1}^{\ell} \frac{1}{(2j)!} \right| = \left| \sum_{j=\ell+1}^{k} \frac{1}{(2j)!} \right| = \frac{1}{(2\ell+2)!} + \frac{1}{(2\ell+4)!} + \cdots + \frac{1}{(2k)!} \\
&= \frac{1}{(2\ell)!} \left(\frac{1}{(2\ell+1)(2\ell+2)} + \frac{1}{(2\ell+1)(2\ell+2)(2\ell+3)(2\ell+4)} + \cdots + \frac{1}{(2\ell+1)(2\ell+2)\cdots(2k-1)(2k)} \right) \\
&\leq \frac{1}{(2\ell)!} \left(\frac{1}{(2\ell+1)(2\ell+2)} + \frac{1}{(2\ell+2)(2\ell+3)} + \cdots + \frac{1}{(\ell+k)(\ell+k+1)} \right) \\
&= \frac{1}{(2\ell)!} \left(\left[\frac{1}{2\ell+1} - \frac{1}{2\ell+2} \right] + \left[\frac{1}{2\ell+2} - \frac{1}{2\ell+3} \right] + \cdots + \left[\frac{1}{\ell+k} - \frac{1}{\ell+k+1} \right] \right) \\
&= \frac{1}{(2\ell)!} \left(\frac{1}{2\ell+1} - \frac{1}{\ell+k+1} \right) \leq \frac{1}{2\ell} \leq \frac{1}{N} < \varepsilon. \quad \square
\end{aligned}$$

From Midterm Given November 17, 2004.

(1.) *Let* $f(x) = (x-1)^2$. *Using the definition of differentiable directly, show that* f *is differentiable at* $a = 4$.

A function is differentiable at a if the limit exists and equals the derivative: $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$. Let $a = 4$ and $f(x) = (x-1)^2$. For $x \neq 4$, the difference quotient equals

$$\frac{f(x)-f(a)}{x-a} = \frac{(x-1)^2 - (4-1)^2}{x-4} = \frac{x^2 - 2x + 1 - 9}{x-4} = \frac{(x-4)(x+2)}{x-4} = x + 2$$

which tends to $4 + 2 = 6$ as $x \to 4$ by the sum theorem for limits. Since the limit exists and equals 6, we conclude that f is differentiable at 4 and $f'(4) = 6$. $\square$

(2.) *Prove that* $\lim\limits_{x\to\infty} \left(x^3 - 5x - 6\right) = \infty$.

$\lim\limits_{x\to\infty} f(x) = \infty$ means for every $M \in \mathbf{R}$ there is a $X_0 \in \mathbf{R}$ so that for every $x \in \mathbf{R}$ such that $x > X_0$, we have $f(x) > M$.

Choose $M \in \mathbf{R}$. Let $X_0 = \max\{4, (3|M|)^{1/3}\}$. Choose $x \in \mathbf{R}$ such that $x > X_0$. Since $x > 4$ it follows that $x^3 > 18$ which implies $\frac{1}{3}x^3 > 6$. Since $x > 4$ it also follows that $x^2 > 15$ which implies $\frac{1}{3}x^2 > 5$. Finally, since $x > (3|M|)^{1/3} \geq 0$ we get

$$\begin{aligned}
f(x) = x^3 - 5x - 6 &= \tfrac{1}{3}x^3 + \left(\tfrac{1}{3}x^2 - 5\right)x + \tfrac{1}{3}x^3 - 6 \\
&> \tfrac{1}{3}\left((3|M|)^{1/3}\right)^3 + \tfrac{1}{3}(5-5)x + (6-6) = |M| + 0 + 0 \geq M. \quad \square
\end{aligned}$$

Alternately, for $x > 0$, use the function version of Theorem 2.15(iii) (see p. 69): If there are numbers $X_1, y_0 > 0$ such that $b(x) \geq y_0$ for all $x > X_0$ and $u(x) \to \infty$ as $x \to \infty$ then $b(x) \cdot u(x) \to \infty$ as $x \to \infty$. Hence, as $x \to \infty$,

$$x^3 - 6x - 6 = \left(\frac{x^3 - 5x - 6}{x^3}\right) \cdot x^3 = \left(1 - \frac{5}{x^2}x - \frac{6}{x^3}\right) \cdot x^3 = b(x) \cdot u(x) \to \infty.$$

The conditions on $b(x)$ and $u(x)$ can be seen as follows: By choosing $X_1 = 10$, we see that $x > X_1$ implies $b(x) = 1 - \frac{5}{x^2} - \frac{6}{x^3} > 1 - 0.05 - 0.006 > 0.9$ so $y_0 = 0.9$. Similarly for $x > X_1 = 10$ we see that $u(x) = x^3 = x^2 \cdot x > 100x > x$ which tends to infinity by assumption, so $u(x) \to \infty$ (see Prob. 71[7a].) $\square$

(3.) *Show that the set E is infinite, where $E = \{x \in \mathbf{R} : x\cos x = 7\sin x\}$.*

The function $h(x) = x\cos x - 7\sin x$ is continuous on $\mathbf{R}$ since it is the difference of products of the continuous functions x, $\sin x$ and $\cos x$. Let $x_k = 2\pi k$ and $y_k = 2\pi k + \pi$. Observe that for $k \in \mathbf{N}$, $x_k < y_k < x_{k+1}$ for all k. Now $h(x_k) = (2\pi k)\cos(2\pi k) - 7\sin(2\pi k) = 2\pi k > 0$ and $h(y_k) = (2\pi k + \pi)\cos(2\pi k + \pi) - 7\sin(2\pi k + \pi) = -(2\pi k + \pi) < 0$. Thus for each k, h is a continuous function on the closed bounded interval $[x_k, y_k]$ such that $h(x_k) > 0 > h(y_k)$. By the Intermediate Value Theorem, there is a $z_k \in (x_k, y_k)$ such that $h(z_k) = 0$ so $z_k \in E$. Now, as the z_k's are all distinct, E is infinite because it contains the countably infinite set $\{z_k : k \in \mathbf{N}\}$. To see the distinctness, suppose $k, \ell \in \mathbf{N}$ such that $k \ne \ell$. We may assume $k < \ell$. Then $z_k < y_k < x_{k+1} < y_{k+1} < \cdots < x_{\ell-1} < y_{\ell-1} < x_\ell < z_\ell$ so $z_k \ne z_\ell$. (Of course there are more zeros, such as the ones from the increasing parts of h.) $\square$

(4.) *Let $f(x) = \dfrac{x^2}{1+|x|}$. Show that f is uniformly continuous on $\mathbf{R}$.*

f is uniformly continuous on $\mathbf{R}$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ so that for every $x, y \in \mathbf{R}$ such that $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

Choose $\varepsilon > 0$. Let $\delta = \varepsilon$. Choose $x, y \in \mathbf{R}$ such that $|x - y| < \delta$. Then since

$$|x + y| + |x|\,|y| \le |x| + |y| + |x|\,|y| \le 1 + |x| + |y| + |x|\,|y| = (1 + |x|)(1 + |y|)$$

we get $|f(x) - f(y)| =$

$$\begin{aligned}
&= \left|\frac{x^2}{1+|x|} - \frac{y^2}{1+|y|}\right| = \left|\frac{x^2(1+|y|) - y^2(1+|x|)}{(1+|x|)(1+|y|)}\right| = \left|\frac{(x^2 - y^2) + (x^2|y| - y^2|x|)}{(1+|x|)(1+|y|)}\right| \\
&= \left|\frac{(x-y)(x+y) + (|x|^2|y| - |y|^2|x|)}{(1+|x|)(1+|y|)}\right| \le \frac{|x-y|\,|x+y| + |x||y|\,\big||x| - |y|\big|}{(1+|x|)(1+|y|)} \\
&\le \frac{|x-y|\,|x+y| + |x||y|\,|x-y|}{(1+|x|)(1+|y|)} \le \frac{|x+y| + |x||y|}{(1+|x|)(1+|y|)}|x-y| \le |x-y| < \delta = \varepsilon. \quad \square
\end{aligned}$$

(5.) *Determine whether the statements are true or false. If the statement is true, give the reason. If the statement is false, provide a counterexample.*

(a.) **Statement.** *Let $f : \mathbf{R} \to \mathbf{R}$ be a bounded function. Then there is at least one point $a \in \mathbf{R}$ such that f is continuous at a.*

FALSE. Let $f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q}, \\ 0, & \text{if } x \in \mathbf{R}\backslash\mathbf{Q}. \end{cases}$ *Then $|f(x)| \le 1$ for all x so f is bounded, but f is not continuous anywhere.*

(b.) **Statement.** *Let $f : \mathbf{R} \to \mathbf{R}$ be continuous at 0. Then $g(x) = f(x^2)$ is differentiable at $x = 0$.*

FALSE. Let $f(x) = \sqrt{|x|}$. Then $f(x)$ is (uniformly) continuous on $\mathbf{R}$ (so continuous at 0,) but $f(x^2) = \sqrt{|x^2|} = |x|$ which is not differentiable at 0.

(c.) **Statement.** *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $x = 0$. Let $a_k = f\left(\frac{1}{k}\right)$. Then $\lim_{k\to\infty} a_k$ exists.*

TRUE. f differentiable at 0 implies that f is continuous at 0. In other words, the limit exists and $\lim_{x\to 0} f(x) = f(0)$. But by the sequential characterization of continuity at zero, there is an $L \in \mathbf{R}$ such that any sequence $x_k \ne 0$ such that $x_k \to 0$ as $k \to \infty$, then $\lim_{k\to\infty} f(x_k) = L$ where $L = f(0)$. In particular, $x_k = \frac{1}{k}$ is such a sequence, so $\lim_{k\to\infty} f\left(\frac{1}{k}\right)$ exists and equals $f(0)$.