Homework for Math 3210 §2, Fall 2009

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Please read the relevant sections in the text *Foundations of Analysis* by Joseph L. Taylor and *Math 3210 Supplemental Notes: Basic Logic Concepts* by Anne Roberts.

Please hand in problems A1 – A4 on Friday, August 28.

A1. Truth table. Construct a truth table for the following statement.

$$[P \wedge (P \Rightarrow Q)] \Rightarrow Q.$$

A2. Equivalent Statements. Verify using truth tables that

$$P \wedge [\sim (Q \wedge R)]$$

is equivalent to

$$(P \wedge [\sim Q]) \vee (P \wedge [\sim R]).$$

A3. Quantified Statements. Determine the truth value of each statement assuming that x, y, z are real numbers.

$$(\exists x)(\forall y)(\exists z)(x + y = z);$$
$$(\exists x)(\forall y)(\forall z)(x + y = z);$$
$$(\forall x)(\forall y)(\exists z)[(z > y) \Rightarrow (z > x + y)];$$
$$(\forall x)(\exists y)(\forall z)[(z > y) \Rightarrow (z > x + y)].$$

A4. Negate and Interpret. Write formally, with quantifiers in the right order. Negate the sentence and interpret.

"Everybody doesn't like something but nobody doesn't like Sara Lee."

Please hand in problems B on Friday, September 4.

B. Problems from Taylor's *Foundations of Analysis.*

7[2, 4, 8, 10, 11, 13, 14(part c. only)]

Please hand in problems C1 – C2 on Friday, September 11.

C1. Problems from Taylor's *Foundations of Analysis.*

14[2, 6, 8], 20[7]

C2. In a commutative ring $(R, +, \times)$, show that

$$-(-x) = x \qquad \text{for all } x \in R.$$

C3. (postponed until next week)

Please hand in problems D1 – D2 on Friday, September 18.

D1. (postponed until next week)

D2. (This was problem C3 from last week.) Assume that the integers $(\mathbb{Z}, +, \times)$ with the usual addition and multiplication satisfy the axioms of a commutative ring. Show that the rational numbers $(\mathbb{Q}, +, \times)$ as given by the construction on pp. 17–18 satisfy the distributive axiom "**D**" on p. 16.

Please hand in problems E1 – E4 on Friday, September 25.

E1. (This was problem D1 from last week.) Problems from Taylor's *Foundations of Analysis.*

20[9, 11], 26[1, 2]

E2. Prove that if $x < y$ are two real numbers then there is a rational number p and an irrational number q such that $x < p < q < y$.

E3. The Well Ordering Principle for the natural numbers says that every nonempty subset $S \subset \mathbb{N}$ has a least element. It is a consequence of the Peano axioms (see 15[17]). Show that for every real number $x > 1$ there is a natural number $n \in \mathbb{N}$ such that $n < x \le n + 1$.

E4. Find the supremem and infimum of the real set $E = \left\{ \dfrac{n^2 - 5n + 26}{n^2 - 6n + 10} : n \in \mathbb{N} \right\}$.

Please hand in problems F1 – F3 on Friday, October 2.

F1. Problems from Taylor's *Foundations of Analysis.*

33[8a, 9c], 41[8], 44[8, 9, 10]

F2. Prove that if a, b, x, y are real numbers that satisfy the inequalities

$$|x - a| < 1, \qquad |y - b| < 2, \qquad |a - b| > 7$$

then $|x - y| > 4$.

F3. Suppose $\{a_n\}$ is a convergent sequence. Suppose there are real numbers N and c such that $a_n < c$ whenever $n > N$. Show that

$$\lim_{n\to\infty} a_n \le c.$$

Please hand in problems G1 on Friday, October 9.

G1. Problems from Taylor's *Foundations of Analysis.*

48[1, 8], 53[3, 4, 9], 58[1, 2, 6]

Please hand in problems H1 on Friday, October 23.

H1. Problems from Taylor's *Foundations of Analysis.*

58[3†, 11, 12], 70[4, 8, 10, 11]

(†.) You may assume that the collection of intervals is *countable.* That is, there are extended real numbers $a_i < b_i$ for each $i \in \mathbb{N}$ such that $I \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$. The way that the problem is stated, the collection could be uncountable so that the intervals cannot be listed (put in one-to-one correspondence with $\mathbb{N}$.)

Please hand in problems I1–I2 on Friday, October 30.

I1. Problems from Taylor's *Foundations of Analysis.*

75[2, 5, 10, 12], 80[3, 4, 7, 8]

I2. Let $(a, b) \subset \mathbb{R}$ be an open interval, $c \in (a, b)$ and let $f : (a, b) \to \mathbb{R}$. Show that $f(x)$ is continuous at $c \in (a, b)$ if and only if

$$f(c) = \lim_{x \to c} f(x).$$

The limit of a function is defined similarly to the limit of a sequence. Note that the limit of a function does not involve the value of the function at c.

Definition. Let $c \in [a, b]$ and $f : (a, b) \to \mathbb{R}$. We say that $L \in \mathbb{R}$ is the *limit of* $f(x)$ *as* $x \to c$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - L| < \epsilon \qquad \text{whenever } x \in (a, b),\ x \neq c \text{ and } |x - c| < \delta.$$

To say that the limit of f as $x \to c$ exists and equals L we write $L = \lim_{x \to c} f(x)$.

Please hand in problems J1 on Friday, November 6, 2009.

J1. Problems from Taylor's *Foundations of Analysis.*

85[3, 10], 92[8, 13(a^+ only), 15($f > 0$ and b^- only)], 97[1].

Please hand in problems K1-K2 on Friday, November 13, 2009.

K1. Problems from Taylor's *Foundations of Analysis.*

93[11], 97[4, 9, 11]

K2. Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ and that $f(c) \neq 0$.

(a.) Show that $f(c + h) \neq 0$ for h sufficiently small.

(b.) Using Definition 4.2.1 of the derivative directly, show that $1/f(x)$ is differentiable at c and that

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f^2(c)}.$$

(c.) Use the Product Rule and (b.) to deduce the Quotient Rule 4.2.6 d.

Please hand in problems L1-L2 on Friday, November 20, 2009.

L1. Problems from Taylor's *Foundations of Analysis.*

102[3, 5, 7, 8], 107[1, 2, 11, 12]

L2. (This is part of problem 108[16].) Suppose that $f, g : (0, \infty) \to \mathbb{R}$ are differentiable and that g and g' are never zero. Suppose that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \infty.$$

Show that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

Please hand in problems M1 on Wednesday, November 25, 2009.

M1. Problems from Taylor's *Foundations of Analysis.*

115[3, 4, 8, 10], 122[5].