Math 3210 § 2. **First Midterm Exam** Name: Solutions

Treibergs September 10, 2025.

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1. *Prove that for all $n \in \mathbb{N}$,*

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{n}{2(n+2)}. \tag{1}$$

Proof by induction. For the base case $n = 1$, the equation (1) holds.

$$\text{LHS} = \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{1}{2 \cdot 3} = \frac{1}{6}; \qquad \text{RHS} = \frac{n}{2(n+2)} = \frac{1}{2(3)} = \frac{1}{6}.$$

For the induction case, suppose the equality (1) holds for some n. Then for $n+1$, using the induction hypothesis,

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{(k+1)(k+2)} &= \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} + \sum_{k=n+1}^{n+1} \frac{1}{(k+1)(k+2)} \\
&= \frac{n}{2(n+2)} + \frac{1}{(n+2)(n+3)} \\
&= \frac{1}{n+2}\left[\frac{n}{2} + \frac{1}{n+3}\right] \\
&= \frac{1}{n+2} \cdot \frac{n^3 + 3n + 2}{2(n+3)} \\
&= \frac{1}{n+2} \cdot \frac{(n+2)(n+1)}{2(n+3)} \\
&= \frac{(n+1)}{2[(n+1)+2]}
\end{aligned}$$

which is the statement for $n+1$. Since both the base case and induction case hold, by induction (1) holds for all n.

2. *Recall the axioms of a field $(\mathcal{F}, +, \times)$. For any $x, y, z \in \mathcal{F}$,*

[A1.]	(Commutativity of Addition)	$x + y = y + x$.
[A2.]	(Associativity of Addition)	$x + (y + z) = (x + y) + z$.
[A3.]	(Additive Identity)	$(\exists\, 0 \in \mathcal{F})\ (\forall\, t \in \mathcal{F})\ 0 + t = t$.
[A4.]	(Additive Inverse)	$(\exists\, {-x} \in \mathcal{F})\ x + (-x) = 0$.
[M1.]	(Commutativity of Multiplication)	$xy = yx$.
[M2.]	(Associativity of Multiplication)	$x(yz) = (xy)z$.
[M3.]	(Multiplicative Identity)	$(\exists\, 1 \in \mathcal{F})\ 1 \neq 0$ and $(\forall\, t \in \mathcal{F})\ 1t = t$.
[M4.]	(Multiplicative Inverse)	If $x \neq 0$ then $(\exists\, x^{-1} \in \mathcal{F})\ (x^{-1})x = 1$.
[D.]	(Distributivity)	$x(y + z) = xy + xz$.

Using only the field axioms, show that if $a, b, c \in F$ and $a \neq 0$ then there is at most one solution x to $ax + b = c$. Justify every step of your argument using just the axioms listed here. Do not quote any formulas from the text.

We suppose that there are two solutioms, call them y and z and argue they must be the same number. They both satisfy $ay+b=c$ and $az+b=c$

$ay+b=az+b$	Both expressions equal c, thus they equal each other.
$(ay+b)+(-b)=(az+b)+(-b)$	By the existence of additive inverse (A4), there is $-b$. Post add it to both sides.
$(ay)+(b+(-b))=(az)+(b+(-b))$	By associativity of addition (A3).
$(ay)+0=(az)+0$	By property of additive inverse (A4).
$0+(ay)=0+(az)$	By commutativity of addition (A1).
$ay=az$	By property of additive identity (A3).
$a^{-1}(ay)=a^{-1}(az)$	Since $a\neq 0$, by existence of multiplicative inverse (M4), there is a^{-1}. Pre multiply both sides.
$(a^{-1}a)y=(a^{-1}a)z$	By associativity of multiplication (M2).
$1\cdot y=1\cdot z$	By property of multiplicative inverse (M4).
$y=z$	By property of multiplicative identity (M3).

We conclude that the two solutions are equal. Thus there is at most one solution to this equation.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) Statement: *Let $f:A\to B$ and $E,G\subset A$. Then $f(E\cap G)=f(E)\cap f(G)$.*
   False. Let $A=B=\mathbf{R}$, $f(x)=x^2$, $E=(-2,1]$ and $G=[1,2)$. Then $E\cap G=\emptyset$ so $f(E\cap G)=\emptyset$ but $f(E)=f(G)=[1,4)$ so that $f(E)\cap f(G)=[1,4)$.

   (b) Statement. *Let $(\mathbb{N},s(\cdot))$ be a number system satisfying the Peano Axioms. Then the successor function $s(x)$ is onto.*
   False. The Peano Axiom **N3** "1 is not the successor of an element of $\mathbb{N}$" tells us that 1 is not in the range of s, thus s is not onto.

   (c) Statement. $A=\{x\in\mathbf{R}:(\forall s>1)(\exists t<s)(t\le x<s)\}=(-\infty,1]$.
   True. In terms of set equality $A=\bigcap_{s>1}\bigcup_{t<s}[t,s)=\bigcap_{s>1}(-\infty,s)=(-\infty,1]$.
   We can also argue that $A\subset(-\infty,1]$ and $(-\infty,1]\subset A$.
   For the first, let $y\in A$. Since for all $s>1$ we have there is some $t<s$ so that $t\le y<s$ says $y<s$ for all $s>1$. Thus $y\le 1$ or $y\in(-\infty,1]$. For the second, if $z\in(-\infty,1]$ then $z\le 1$. But this says for all $s>1$ we have $z<s$. By taking $t=x$ there is $t<s$ such that $t\le z<s$. Thus $z\in A$.

4. (a) *Determine whether the cube root of 7 is irrational.*
   $x=\sqrt[3]{7}$ is irrational. Argue by contradiction. Suppose that x were rational

$$x=\frac{p}{q}$$

   where p and q are integers without a common factor. Then $x^3=7$ or

$$p^3=7q^3$$

   Thus $7|p^3$. Since 7 is prime, $7|p$ so $p=7k$ for some integer k. Hence

$$7^3k^3=7q^3$$

or

$$q^3 = 7^2 k^3.$$

Thus $7|q^3$. Since 7 is prime, $7|q$. We have reached a contradiction: 7 is a common factor of both p and q. Thus the statement must have been true: $\sqrt[3]{7}$ is irrational.

(b) *Recall that the rational numbers are defined to be the set of equivalence classes* $\mathbb{Q} = S/\sim$ *where* $S = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0 \right\}$ *is the set of symbols (pairs of integers) and the symbols are equivalent if they represent the same fraction:* $\frac{a}{b} \sim \frac{c}{d}$ *iff* $ad = bc$. *We denote the equivalence class, the "fraction" by* $\left[\frac{a}{b}\right]$ *to distinguish it from a symbol from* S. *Addition and multiplication of rationals, for example, is defined on equivalence classes by*

$$\left[\frac{m}{n}\right] + \left[\frac{r}{t}\right] = \left[\frac{mt + nr}{nt}\right], \qquad \left[\frac{m}{n}\right] \cdot \left[\frac{r}{t}\right] = \left[\frac{mr}{nt}\right].$$

Show that the distributive axiom $x(y+z) = xy + xz$ *holds for all* x, y *and* z *in the rational numbers.*

Choose representatives of the three rational numbers

$$x = \left[\frac{p}{q}\right], \qquad y = \left[\frac{a}{b}\right], \qquad z = \left[\frac{r}{s}\right]$$

were p, q, a, b, r, s are integers where q, b, s are nonzero. Then using the definitions of addition and multiplication

$$\begin{aligned} x(y+z) &= \left[\frac{p}{q}\right]\left(\left[\frac{a}{b}\right] + \left[\frac{r}{s}\right]\right) = \left[\frac{p}{q}\right]\left(\left[\frac{as+br}{bs}\right]\right) = \left[\frac{p(as+br)}{q(bs)}\right] = \left[\frac{qp(as+br)}{q^2(bs)}\right] \\ &= \left[\frac{(pa)(qs) + (qb)(pr)}{(qb)(qs)}\right] = \left[\frac{pa}{qb}\right] + \left[\frac{pr}{qs}\right] = \left[\frac{p}{q}\right]\left[\frac{a}{b}\right] + \left[\frac{p}{q}\right]\left[\frac{r}{s}\right] = xy + xz \end{aligned}$$

5. *A subset of the real numbers is given by* $E = \left\{ \frac{n^2}{2n^2 - 1} : n \in \mathbb{N} \right\}$.

Find the greatest lower bound $\ell = \operatorname{glb} E$ *and prove your result.*

We claim that the greatest lower bound is $\ell = \frac{1}{2}$. To see that it is a lower bound, for every $n \in \mathbb{N}$ we have

$$\frac{n^2}{2n^2 - 1} \geq \frac{n^2 - \frac{1}{2}}{2n^2 - 1} = \frac{1}{2}.$$

To see that there is no greater lower bound, choose $m > \frac{1}{2}$ to show m is not a lower bound. By the Archimedean Property of the reals, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \sqrt{2m - 1}.$$

For such n_0 we have

$$\begin{aligned} \frac{n_0^2}{2n_0^2 - 1} &= \frac{n_0^2 - \frac{1}{2} + \frac{1}{2}}{2n_0^2 - 1} = \frac{1}{2} + \frac{1}{2(2n_0^2 - 1)} \\ &\leq \frac{1}{2} + \frac{1}{2(2n_0^2 - n_0^2)} = \frac{1}{2} + \frac{1}{2n_0^2} < \frac{1}{2} + \left(m - \frac{1}{2}\right) = m. \end{aligned}$$

Thus an element of E is less than m so that m is not a lower bound. Thus $\ell = \frac{1}{2}$ is the greatest lower bound.