Math 3210 § 1. **First Midterm Exam** Name: Solutions
Treibergs February 1, 2023.

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1. *Prove that for all $n \in \mathbb{N}$,*
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \tag{1}$$
Argue by induction.

Base case: $n = 1$. Compute both sides.
$$\text{Lhs} = \sum_{k=1}^{1} k^3 = 1^3 = 1, \qquad \text{Rhs} = \frac{1^2(1+1)^2}{4} = 1.$$
The left side equals the right side, so the base case holds.

Induction case: Assume for some n the induction hypothesis holds
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$
For $n+1$, by the induction hypothesis,
$$\sum_{k=1}^{n+1} k^3 = (n+1)^3 + \sum_{k=1}^{n} k^3 = (n+1)^3 + \frac{n^2(n+1)^2}{4} = (n+1)^2\left((n+1) + \frac{n^2}{4}\right)$$
$$= \frac{(n+1)^2}{4}(4 + 4n + n^2) = \frac{(n+1)^2(n+2)^2}{4} = \frac{(n+1)^2[(n+1)+1]^2}{4}$$
which proves the equation (1) for $n+1$. Hence by induction, (1) holds for all $n \in \mathbb{N}$.

2. *Recall the axioms of a field $(\mathcal{F}, +, \times)$. For any $x, y, z \in \mathcal{F}$,*

[A1.]	(Commutativity of Addition)	$x + y = y + x$.
[A2.]	(Associativity of Addition)	$x + (y + z) = (x + y) + z$.
[A3.]	(Additive Identity)	$(\exists\, 0 \in \mathcal{F})\,(\forall\, t \in \mathcal{F})\ 0 + t = t$.
[A4.]	(Additive Inverse)	$(\exists\, {-x} \in \mathcal{F})\ x + (-x) = 0$.
[M1.]	(Commutativity of Multiplication)	$xy = yx$.
[M2.]	(Associativity of Multiplication)	$x(yz) = (xy)z$.
[M3.]	(Multiplicative Identity)	$(\exists\, 1 \in \mathcal{F})\ 1 \neq 0$ and $(\forall\, t \in \mathcal{F})\ 1t = t$.
[M4.]	(Multiplicative Inverse)	If $x \neq 0$ then $(\exists\, x^{-1} \in \mathcal{F})\ (x^{-1})x = 1$.
[D.]	(Distributivity)	$x(y + z) = xy + xz$.

Using only the field axioms, show that for $x + y = x$ implies $y = 0$. Justify every step of your argument using just the axioms listed here. Do not quote any formulas from the text.

$x + y = x$	Assumption.
$(-x) + (x + y) = (-x) + x$	By [A4] there is $-x$. Pre-add to both sides.
$((-x) + x) + y = (-x) + x$	[A2]
$(x + (-x)) + y = x + (-x)$	[A1]
$0 + y = 0$	[A4]
$y = 0$	[A3]

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Let $f : A \to B$ and $E \subset A$. Then $f^{-1}(f(E)) = E$.*
   FALSE. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2$ and $E = [2, 3)$. Then $f(E) = [4, 9)$ and $f^{-1}(f(E)) = (-3, -2] \cup [2, 3)$.

   (b) STATEMENT: *Let $(\mathbb{N}, s(\cdot))$ be a number system satisfying the Peano Axioms. Then the successor function $s(x)$ is one-to-one.*
   TRUE. The fourth Peano axiom says if two elements of $\mathbb{N}$ have the same successor, then they are equal. That is the definition of one-to-one.

   (c) STATEMENT: $\left(1 + \frac{1}{n}\right)^n \ge 2$ *for all natural numbers n.*
   TRUE. By the binomial formula,

$$
\begin{aligned}
\left(\frac{1}{n} + 1\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k 1^{n-k} \\
&= \binom{n}{0} \left(\frac{1}{n}\right)^0 1^n + \binom{n}{1} \left(\frac{1}{n}\right)^1 1^{n-1} + \sum_{k=2}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k 1^{n-k} \\
&\ge 1 \cdot 1 \cdot 1 + n \cdot \left(\frac{1}{n}\right) \cdot 1 + 0 = 2.
\end{aligned}
$$

4. *Recall that the rational numbers are defined to be the set of equivalence classes $\mathbb{Q} = S/\sim$ where $S = \left\{\frac{a}{b} : a, b \in \mathbb{Z},\ b \ne 0\right\}$ is the set of symbols (pairs of integers) and the symbols are equivalent if they represent the same fraction: $\frac{a}{b} \sim \frac{c}{d}$ iff $ad = bc$. We denote the equivalence class, the "fraction" by $\left[\frac{a}{b}\right]$ to distinguish it from a symbol from S. Addition and multiplication of rationals, for example, is defined on equivalence classes by*

$$\left[\frac{m}{n}\right] + \left[\frac{r}{t}\right] = \left[\frac{mt + nr}{nt}\right], \qquad \left[\frac{m}{n}\right] \cdot \left[\frac{r}{t}\right] = \left[\frac{mr}{nt}\right].$$

   (a) *Show that the distributive axiom $x(y + z) = xy + xz$ holds for all x, y and z in the rational numbers.*
   Suppose the rational numbers are represented by fractions

$$x = \left[\frac{a}{b}\right], \qquad y = \left[\frac{c}{d}\right], \qquad z = \left[\frac{e}{f}\right].$$

   We use the formulae for addition and multiplication and arithmetic in $\mathbb{Z}$,

$$x(y + z) = \left[\frac{a}{b}\right] \left(\left[\frac{c}{d}\right] + \left[\frac{e}{f}\right]\right) = \left[\frac{a}{b}\right] \left[\frac{cf + de}{df}\right] = \left[\frac{a(cf + de)}{b(df)}\right] = \left[\frac{acf + ade}{bdf}\right]$$

   But this equals

$$\left[\frac{abcf + abde}{b^2 df}\right]$$

   because the the symbols are equivalent

$$(acf + ade)(b^2 df) = (bdf)(abcf + abde).$$

Thus

$$x(y+z) = \left[\frac{abcf + abde}{b^2df}\right] = \left[\frac{(ac)(bf) + (bd)(ae)}{(bd)(bf)}\right]$$
$$= \left[\frac{ac}{bd}\right] + \left[\frac{ae}{bf}\right] = \left[\frac{a}{b}\right]\left[\frac{c}{d}\right] + \left[\frac{a}{b}\right]\left[\frac{e}{f}\right] = xy + xz.$$

(b) *How is $x > y$ defined in the rational numbers?*

$x > y$ in an ordered field means $x \geq y$ and $x \neq y$ which in turn means $x - y \geq 0$ and $x \neq y$. If the rational numbers are represented by fractions

$$x = \left[\frac{a}{b}\right], \qquad y = \left[\frac{c}{d}\right], \qquad z = \left[\frac{e}{f}\right]$$

then $z \geq 0$ means $ef \geq 0$ so that $x - y \geq 0$ becomes

$$x - y = \left[\frac{a}{b}\right] - \left[\frac{c}{d}\right] = \left[\frac{ad - be}{bd}\right] \geq 0$$

or $(ad - be)(bd) \geq 0$ and $x \neq y$ becomes $ad \neq bc$.

Equivalently, WLOG we may assume positive denominators $b > 0$, $d > 0$ and $f > 0$. In this case $z \geq 0$ is $e \geq 0$, $x - y \geq 0$ is $ad - be \geq 0$ and $x \neq y$ is $ad \neq bc$.

5. *For ϵ, δ real, let E be the given subset of the real numbers. Determine E. Prove that your set equals E above.*

$$E = \{x \in \mathbf{R} : (\forall \epsilon > 0)\,(\exists \delta > 0)\;(-\delta < x < \epsilon)\}$$

We may determine E by writing the set as an intersection of a union of intervals

$$E = \bigcap_{\epsilon > 0} \bigcup_{\delta > 0} (-\delta, \epsilon) = \bigcap_{\epsilon > 0} (-\infty, \epsilon) = (-\infty, 0].$$

To prove equality we first argue $(-\infty, 0] \subset E$. Choose $x \in (-\infty, 0]$. Hence $x \leq 0$. To show that $x \in E$, choose $\epsilon > 0$. Let $\delta = -x + 1 > 0$. For these, $-\delta = x - 1 < x \leq 0 < \epsilon$. Thus $x \in E$.

Next we argue $E \subset (-\infty, 0]$ which is to say if $x \in E$ then $x \in (-\infty, 0]$. By contrapositive, this is equivalent to if $x \notin (-\infty, 0]$ then $x \notin E$. But $x \notin (-\infty, 0]$ implies $x > 0$. Hence there is an $\epsilon = x > 0$ such that $\epsilon \leq x$. But his implies $x \notin E$, namely

$$\sim (\forall \epsilon > 0)\,(\exists \delta > 0)\;(-\delta < x < \epsilon) \equiv (\exists \epsilon > 0)\,(\forall \delta > 0)\;(x \leq -\delta \text{ or } \epsilon \leq x).$$

The statement $(\forall \delta > 0)\;(x \leq -\delta \text{ or } \epsilon \leq x)$ is true because $\epsilon \leq x$ for this ϵ.