Worksheets for Math 3210 §2, Spring 2020

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In this CoViD-19 period of distance leaarning, the commentaries play the roles of daily lectures. Worksheets are daily assignments that give you the opportunity to show that you are participating. They are intended to relate to the the topic being discussed in the commentary, and should be easier than homework. Worksheets are due the day of the next class. Your worksheets will be scored, but as far as the grading goes, they will have weight zero in your final grade. For your convenience, the worksheet assignments are listed here.

March 20. (Postponed from Earthquake Wednesday.) State and prove a theorem that says that if two functions are continuous at a point a then their quotient is also continuous at a.

March 23. Let $n \in N$. Show that every positive real number a has a positive n-th root.

March 24. Find a function that takes a closed unit interval to a closed unit interval which has an inverse but the inverse is not continuous.

March 27. Prove $f(x) = \sqrt{1+x^2}$ is uniformly continuous on $\mathbb{R}$.

March 30. Let $f(x) = \dfrac{1}{1+x^2}$ be functions on $\mathbf{R}$. Determine whether $\{f_n\}$ converges uniformly, converges pointwise but not uniformly or does not converge on $\mathbf{R}$, and prove your answer.

March 31. Let $g(x) = \dfrac{1}{f^2(x)}$ where $f : R \to \mathbf{R}$ is differentiable at $a\imath\mathbf{R}$ and satisfies $f(a) \neq 0$. Using just the definition of differentiable, determine whether g is differentiable at a, and if it is, what is $g?(a)$?

April 1. Suppose $f : \to \mathbf{R}$ is differentiable and that there is a positive number m such that $f?(x) \geq m$ for all x. Prove that

$$f(x) \geq f(0) + mx, \text{ if } x \geq 0 \quad \text{and} \quad f(x) \leq f(0) + mx \text{ if } x \leq 0.$$

Use these inequalities to prove $f(\mathbf{R}) = \mathbf{R}$.

April 3. Prove that if $r > 1$ and $x > 1$ then $\ln x \leq \dfrac{x^r - 1}{r}$.

April 6. The Heaviside Function is defined by

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Using just the definition of integrable, prove that the Heaviside Function is integrable on $[a, b]$, where $a < 0 < b$. What is the value of the integral?

April 7. Let $I = [a, b]$ be a closed and bounded region and $E \subset I$ a finite set. Show that χ_E is integrable on I. What is the value of the integral?

April 8. Show that f is integrable on $[0, 1]$ where

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

April 10. The signum function is defined by

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Its derivative is zero everywhere but $x = 0$. Its derivative is integrable on $[-1, 1]$ with integral zero. However $\operatorname{sgn}(1) - \operatorname{sgn}(-1) = 1 - (-1) = 2$. This seems to contradict the First Fundamental Theorem of Calculus. Explain why it does not.

April 13. Let $f : [a, b] \to \mathbf{R}$ be continuous. Show that there is a $c \in (a, b)$ such that

$$\int_a^b f(t)\, dt = f(c)(b - a),$$

[Hint: consider $F(x) = \int_a^x f(t)\, dt$.]

April 14. Determine whether the improper integral converges

$$I = \int_1^\infty \frac{\ln x}{1 + x^3}\, dx.$$

April 15. Show that if $\sum_{i=1}^\infty a_i$ converges and $\sum_{i=1}^\infty b_i$ converges absolutely then $\sum_{i=1}^\infty a_i b_i$ converges absolutely.

April 17. Determine whether the series is convergent, conditionally convergent or divergent.

$$\sum_{k=1}^\infty (-1)^{k+1} \left(\sqrt{k+1} - \sqrt{k} \right).$$

April 20. Show that the series is conditionally convergent and that the Cauchy Product of the series with itself is not convergent.

$$\sum_{k=0}^\infty \frac{(-1)^k}{\sqrt{k+1}}$$