Math 3160 § 1. **Final Exam** Name: Practice Problems
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Half of the final will be comprehensive. The other half will focus on material since the last midterm exam. These practice problems cover this last third of the course.

1. *Let C be any closed contour, described in the positive sense in the z plane, and write*

$$g(z) = \int_C \frac{s^3 + 2s}{(s-z)^3}\, ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and $g(z) = 0$ if z is outside.

See text problem 170[4]. In case z is outside C, then because the function

$$h(s) = \frac{s^3 + 2s}{(s-z)^3}$$

is analytic on and inside C, by the Cauchy-Goursat theorem we have

$$g(z) = \int_C h(s)\, ds = 0.$$

If z is inside C then we may apply the extended Cauchy Integral Formula, namely

$$\frac{2!\, g(z)}{2\pi i} = \frac{2!}{2\pi i}\int_C \frac{k(s)}{(s-z)^3}\, ds = k''(z)$$

where

$$k(s) = s^3 + 2s.$$

Thus we find

$$g(z) = \pi i k''(z) = \pi i (6z).$$

2. *Show for any real constant a,*

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\, d\theta = \pi.$$

See text problem 171[7]. Let C be the unit circle $|z| = 1$ with center $z_0 = 0$ and positive orientation. We first observe from the Cauchy Integral Formula applied to the entire function e^{az} that at $z_0 = 0$,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i\, e^{az_0} = 2\pi i.$$

Rewriting the integral in terms of $z = e^{i\theta} = \cos\theta + i\sin\theta$ for $-\pi \le \theta \le \pi$,

$$2\pi i = \int_{-\pi}^{\pi} \frac{e^{a\cos\theta + ia\sin\theta}}{e^{i\theta}}\, i\, e^{i\theta}\, d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}\left(\cos(a\sin\theta) + i\sin(a\sin\theta)\right)\, d\theta.$$

The imaginary part of this equation is

$$2\pi = \int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\, d\theta = 2\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\, d\theta.$$

We have used the fact that the function $h(\theta) = e^{a\cos\theta}\cos(a\sin\theta)$ is even because it satisfies $h(\theta) = h(-\theta)$ so that areas under the integral to the left and right of zero are equal.

3. *Suppose that $f(z)$ is entire and that*

$$\lim_{z\to\infty}\frac{f(z)}{z}=0.$$

prove that f is constant.

We follow the proof of Liouville's theorem. Let $z_0 \in \mathbf{C}$ be any point and let C_R be the circle $|z-z_0|=R$. Since f is entire, the radius R may be as large as you please. Then by Cauchy's Inequality, the derivative

$$|f'(z_0)| \le \frac{M_R}{R}$$

where M_R is the maximum value of $|f(z)|$ for $z \in C_R$. Because C_R is a closed and bounded set and $|f(z)|$ is continuous on $\mathbf{C}$, there is a point $\zeta(R) \in C_R$ such that $|\zeta(R)| = M_R$. By the triangle inequality,

$$\begin{aligned} R+|z_0| = |z_0-\zeta(R)|+|z_0| &\ge |(\zeta(R)-z_0)+z_0| \\ &= |\zeta(R)| \\ &\ge \big||\zeta(R)-z_0|-|z_0|\big| \ge |\zeta(R)-z_0|-|z_0| = R-|z_0| \end{aligned}$$

so that $|\zeta(R)|\to\infty$ as $R\to\infty$. It follows that

$$|f'(z_0)| \le \frac{M_R}{R} = \frac{|f(\zeta(R))|}{R} = \left|\frac{f(\zeta(R))}{\zeta(R)}\right|\cdot\frac{|\zeta(R)|}{R} \le \left|\frac{f(\zeta(R))}{\zeta(R)}\right|\cdot\frac{R+|z_0|}{R} \to 0\cdot 1 = 0$$

as $R\to\infty$ because, by assumption,

$$\lim_{R\to\infty}\frac{f(\zeta(R))}{\zeta(R)}=0.$$

Since z_0 was arbitrary, $f'=0$ on $\mathbf{C}$ so f is constant.

4. *Suppose f and g are continuous on a closed region in the plane bounded by the simple closed contour C and are both analytic inside C. Suppose that $f=g$ on C. Show that $f=g$ on the whole region.*

The function $h(z)=f(z)-g(z)$ continuous in the region and analytic inside. By the maximum modulus principle,

$$|h(z)| \le \max_{z\in C}|h(z)|$$

which is zero because $h(z)=0$ on C. Therefore $f(z)-g(z)=h(z)=0$ for all z inside C.

5. *Find the maximum of $|e^{z^2}|$ on the unit disk.*

By the maximum modulus principle, the maximum occurs on the boundary. Letting $z=e^{i\theta}$ run through all boundary points as $-\pi\le\theta\le\pi$ we have

$$|e^{z^2}| = |e^{e^{2i\theta}}| = |e^{\cos 2\theta + i\sin 2\theta}| = e^{\cos 2\theta}$$

which is maximum when $\cos 2\theta = 1$ at $\theta=0$ and $\theta=\pm\pi$. Thus

$$|e^{z^2}| \le e$$

with equality at $z=\pm 1$.

6. *Let z_0 be a zero of the polynomial of degree $n \ge 1$,*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \qquad (a_n \neq 0).$$

Show that there is a polynomial $Q(z)$ of degree $n-1$ such that

$$P(z) = (z - z_0)Q(z).$$

See the text problem 178[8]. Note that for any z_0,

$$z^k - z_0^k = (z - z_0)(z_0^{k-1} + z_0^{k-2} z + \cdots + z_0 z^{k-2} + z^{k-1}) = (z - z_0) \sum_{j=0}^{k-1} z^j z_0^{k-1-j}.$$

Thus because the constant terms cancel, switching the order of summation,

$$\begin{aligned}
P(z) - P(z_0) &= \sum_{k=1}^{n} a_k z^k - \sum_{k=1}^{n} a_k z_0^k \\
&= \sum_{k=1}^{n} a_k \left(z^k - z_0^k\right) \\
&= \sum_{k=1}^{n} a_k (z - z_0) \left(\sum_{j=0}^{k-1} z^j z_0^{k-1-j} \right) \\
&= (z - z_0) \sum_{k=1}^{n} \sum_{j=0}^{k-1} a_k z^j z_0^{k-1-j} \\
&= (z - z_0) \sum_{j=0}^{n-1} \left(\sum_{k=j+1}^{n} a_k z_0^{k-1-j} \right) z^j \\
&= (z - z_0) Q(z)
\end{aligned}$$

where $Q(z)$ is of degree $n-1$ since its z^{n-1} coefficient is $a_n \neq 0$. If z_0 is a zero, then $P(z_0) = 0$ and the result follows.

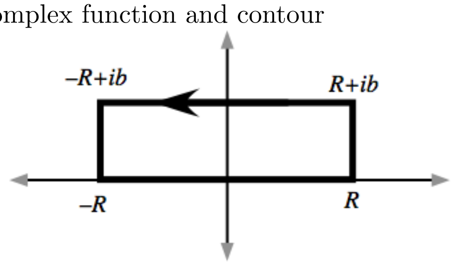
7. *Show that if $b > 0$ then*

$$\mathcal{I} = \int_0^\infty e^{-x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$

See the text problem 159[4]. The improper integral is computed from the limit

$$\lim_{R \to \infty} \int_0^R e^{-x^2} \cos 2bx \, dx.$$

The idea is to find a complex function and contour



where the desired integral appears as the real or imaginary part of

$$I = \int_C f(z)\,dz.$$

The contour and function that works is the rectangle C whose sides are segments from R to $R + bi$ to $-R + bi$ to $-R$ and back to R, and $f(z) = \exp(-z^2)$.

Along the bottom, $z = t$ where t runs from $-R$ to R

$$I_1 = \int_R^R e^{-t^2}\,dt.$$

The limit of integral as $R \to \infty$ is

$$L = \lim_{R\to\infty} I_1 = \int_{-\infty}^{\infty} e^{-t^2}\,dt$$

This is done using a trick you saw in advanced calculus. The integral over the plane is computed using polar coordinates.

$$\begin{aligned}
L^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \iint_{\mathbf{R}^2} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-\rho^2}\rho\,d\rho\,d\theta \\
&= 2\pi\int_0^{\infty} e^{-\rho^2}\rho\,d\rho \\
&= \pi\left[-e^{-\rho^2}\right]_0^{\infty} = \pi.
\end{aligned}$$

The integral along the top $z(t) = ib - t$ where t runs from $-R$ to R. Then $z(t)^2 = -b^2 - 2ibt + t^2$ so that

$$I_3 = -\int_{-R}^{R} e^{b^2+2ibt-t^2}\,dt = -e^{b^2}\int_{-R}^{R} e^{-t^2}\left(\cos 2bt + i\sin 2bt\right)\,dt = -2e^{b^2}\int_0^R e^{-t^2}\cos 2bt\,dt$$

since the real part is the integral of an even function and the imaginary part is the integral of an odd function. Thus the limit is a multiple of the desired integral

$$\lim_{R\to\infty} I_3 = -2e^{b^2}\mathcal{I}.$$

Along the right side of C, $z(t) = R + ti$ where $0 \le t \le b$ so that

$$I_2 = \int_0^b e^{-R^2-2Rti+t^2}\,i\,dt.$$

which is bounded by

$$|I_2| \le \int_0^b \left|e^{-R^2-2Rti+t^2}\,i\right|\,dt = e^{-R^2}\int_0^b e^{t^2}\,dt.$$

which tends to zero as $R \to \infty$. Similarly, along the left side, $z(t) = -R - ti$ where $-b \le t \le 0$ so that

$$I_4 = \int_{-b}^{0} e^{-R^2+2Rti+t^2}\,i\,dt.$$

which is bounded by

$$|I_4| = \int_0^b \left| e^{-R^2+2Rti+t^2}\, i \right| dt \le e^{-R^2} \int_{-b}^0 e^{t^2}\, dt.$$

which tends to zero as $R \to \infty$. Finally, by the Cauchy-Goursat Theorem,

$$0 = \int_C f(z)\, dz = I_1 + I_2 + I_3 + I_4.$$

Taking the limit of theis equation as $R \to \infty$ yields

$$0 = \sqrt{\pi} + 0 - 2e^{b^2}\mathcal{I} + 0$$

so that

$$\mathcal{I} = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

as desired.

8. *Determine whether the sequence converges and if so, prove it using the definition of the convergence.*

$$\lim_{n\to\infty} \frac{n^2 + in - 3}{(n+i)^2}$$

The convergence is implied by the existence of complex limit

$$L = \lim_{z\to\infty} \frac{z^2 + iz - 3}{(z+i)^2} = \lim_{z\to\infty} \frac{z^2 + iz - 3}{z^2 + 2iz - 1} = \lim_{z\to\infty} \frac{1 + \frac{i}{z} - \frac{3}{z^2}}{1 + \frac{2i}{z} - \frac{1}{z^2}} = \frac{1+0+0}{1+0-0} = 1.$$

The proof is almost the same as in the complex limit. If $L \in \mathbf{C}$, we say that $z_n \to L$ as $n \to \infty$ if for every $\epsilon > 0$ there is an $n_0 \in \mathbf{N}$ such that

$$|z_n - L| < \epsilon \qquad \text{whenever} \qquad n > n_0.$$

Choose $\epsilon > 0$. Let $n_0 \in \mathbf{N}$ be an integer so large that $n_0 > \dfrac{8}{\epsilon} + 2$. Then if $n \in \mathbf{N}$ such that $n > n_0$ we have $n > 2$ so by the triangle inequalities,

$$\begin{aligned} |z_n - L| &= \left| \frac{n^2 + in - 3}{(n+i)^2} - 1 \right| = \left| \frac{n^2 + in - 3 - (n^2 + 2ni - 1)}{(n+i)^2} \right| = \frac{|-in-2|}{|n+i|^2} \\ &\le \frac{|n| + 2}{\big||n| - |1|\big|^2} = \frac{n+2}{(n-1)^2} \le \frac{n+n}{(n-\frac{n}{2})^2} = \frac{2n}{n^2/4} = \frac{8}{n} < \frac{8}{n_0} < \frac{8}{8/\epsilon} = \epsilon. \end{aligned}$$

9. *Find the Taylor series with center $z_0 = 3i$ and determine the radius of convergence for the function*

$$f(z) = \frac{2z}{z^2 - 16}.$$

The partial fractions decomposition is

$$f(z) = \frac{2z}{(z+4)(z-4)} = \frac{1}{z+4} + \frac{1}{z-4}.$$

Use the geometric sum

$$\sum_{k=0}^{\infty} w^k = \frac{1}{1-w}$$

which converges exactly when $|w| < 1$. Centering the fractions on $(z - 3i)$ we obtain

$$\begin{aligned}
f(z) &= \frac{1}{4 + 3i + (z - 3i)} - \frac{1}{4 - 3i - (z - 3i)} \\
&= \frac{\dfrac{1}{4+3i}}{1 + \dfrac{z-3i}{4+3i}} - \frac{\dfrac{1}{4-3i}}{1 - \dfrac{z-3i}{4-3i}} \\
&= \frac{1}{4+3i} \sum_{k=0}^{\infty} (-1)^k \left(\frac{z-3i}{4+3i} \right)^k - \frac{1}{4-3i} \sum_{k=0}^{\infty} \left(\frac{z-3i}{4-3i} \right)^k \\
&= -\sum_{k=0}^{\infty} \left(\left(\frac{-1}{4+3i} \right)^{k+1} + \left(\frac{1}{4-3i} \right)^{k+1} \right) (z - 3i)^k
\end{aligned}$$

The series converge if $\left| \dfrac{z-3i}{4+3i} \right| < 1$ and $\left| \dfrac{z-3i}{4-3i} \right| < 1$ which both say $|z - 3i| < 5$. Thus the radius of convergence is at least five. The function $f(z)$ blows up at $z = \pm 4$ which are a distance 5 from $z_0 = 3i$, so that the radius of convergence is at most five, so $R = 5$.

10. *Let C be the unit circle $|z| = 1$ with positive orientation. Find*

$$\int_C \frac{dz}{e^z - 1}.$$

We may use the Residue Theorem. Alternately, we observe that $f(z) = 1/(e^z - 1)$ is singular at $z = 0$. We use the fact that

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$

is the coefficient of the $1/z$ term for the Laurent series of f, provided that f is analytic in some annulus $R_1 < |z| < R_2$ containing C, such that $R_1 < 1 < R_2$. But $e^z = 1$ if and only if $z = 2\pi i n$ for some $n \in \mathbf{Z}$. Hence $f(z)$ is analytic in the annulus $0 < |z| < 2\pi$ containing C. It remains to find the Laurent coefficient.

$$f(z) = \frac{1}{e^z - 1} = \frac{1}{\left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right) - 1} = \frac{1}{z\left(1 + \frac{1}{2}z + \frac{1}{6}z^2 + \cdots\right)}.$$

Doing long division

$$\begin{array}{r|l}
 & 1 - \frac{1}{2}z + \frac{1}{12}z^2 + \cdots \\ \hline
1 + \frac{1}{2}z + \frac{1}{6}z^2 + \cdots & 1 + 0 \cdot z + 0 \cdot z^2 + \cdots \\
 & \underline{1 + \frac{1}{2}z + \frac{1}{6}z^2 + \cdots} \\
 & \quad -\frac{1}{2}z - \frac{1}{6}z^2 + \cdots \\
 & \quad \underline{-\frac{1}{2}z - \frac{1}{4}z^2 + \cdots} \\
 & \qquad\qquad \frac{1}{12}z^2 + \cdots \\
 & \qquad\qquad \underline{\frac{1}{12}z^2 + \cdots} \\
 & \qquad\qquad\ 0 + \cdots
\end{array}$$

we find

$$f(z) = \frac{1}{z}\left(1 - \frac{1}{2}z + \frac{1}{12}z^2 + \cdots\right) = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + \cdots$$

thus $b_1 = 1$. It follows that

$$\int_C \frac{dz}{e^z - 1} = 2\pi i b_1 = 2\pi i.$$

11. *Find the Maclaurin Series up to the z^4 term for*

$$f(z) = \mathrm{Log}(1 + \sin(z))$$

We know that

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + w^4 - w^5 \cdots$$

which converges for $|w| < 1$. By integrating in paths from 0 to ζ in the unit disk $|z| < 1$, we find

$$\mathrm{Log}(1+\zeta) = \int_0^\zeta \frac{dw}{w+1} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

which converges for $|\zeta| < 1$. We shall substitute

$$\zeta = \sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5040} \cdots$$

which converges for all $z \in \mathbf{C}$. We compute up to the z^6 power

$$\begin{aligned}
\zeta^2 &= \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)^2 \\
&= z^2 - 2z \cdot \frac{z^3}{6} + \left(2 \cdot \frac{1}{120} + \frac{1}{6^2}\right) z^6 + \cdots \\
&= z^2 - \frac{z^4}{3} + \frac{2\,z^6}{45} + \cdots \\
\zeta^3 &= \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right) \cdot \left(z^2 - \frac{z^4}{3} + \frac{2\,z^6}{45} + \cdots\right) \\
&= z^3 - \left(\frac{1}{3} + \frac{1}{6}\right) z^5 + 0z^6 \cdots = z^3 - \frac{1}{2}z^5 + 0z^6 \cdots \\
\zeta^4 &= \left(z - \frac{z^3}{6} + \cdots\right) \cdot \left(z^3 - \frac{1}{2}z^5 + 0z^6 \cdots\right) \\
&= z^4 - \left(\frac{1}{2} + \frac{1}{6}\right) z^6 + \cdots = z^4 - \frac{2}{3}z^6 + \cdots \\
\zeta^5 &= \left(z - \frac{z^3}{6} + \cdots\right) \cdot \left(z^4 - \frac{2}{3}z^6 + \cdots\right) = z^5 + 0z^6 \cdots \\
\zeta^6 &= \left(z - \frac{z^3}{6} + \cdots\right) \cdot \left(z^5 + \cdots\right) = z^6 + \cdots
\end{aligned}$$

Substituting the values we find

$$\begin{aligned}
\operatorname{Log}(1+\zeta) &= \zeta - \frac{\zeta^2}{2} + \frac{\zeta^3}{3} - \frac{\zeta^4}{4} + \frac{\zeta^5}{5} - \frac{\zeta^6}{6}\cdots \\
&= \left(z - \frac{z^3}{6} + \frac{z^5}{120} + 0z^6 - \cdots\right) - \frac{1}{2}\left(z^2 - \frac{z^4}{3} + \frac{2\,z^6}{45} + \cdots\right) + \frac{1}{3}\left(z^3 - \frac{1}{2}z^5 + 0z^6\cdots\right) \\
&\quad - \frac{1}{4}\left(z^4 - \frac{2}{3}z^6 + \cdots\right) + \frac{1}{5}\left(z^5 + 0z^6\cdots\right) - \frac{1}{6}\left(z^6\cdots\right) + \cdots \\
&= z - \frac{z^2}{2} + \left(-\frac{1}{6} + \frac{1}{3}\right)z^3 + \left(\frac{1}{2}\cdot\frac{1}{6} - \frac{1}{4}\right)z^4 + \left(\frac{1}{120} - \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{5}\right)z^5 \\
&\quad + \left(-\frac{1}{2}\cdot\frac{2}{45} + \frac{1}{4}\cdot\frac{2}{3} - \frac{1}{6}\right)z^6 + \cdots \\
&= z - \frac{z^2}{2} + \frac{z^3}{6} - \frac{z^4}{6} + \frac{5\,z^5}{120} - \frac{z^6}{45} + \cdots
\end{aligned}$$

12. *Let C be the positively oriented circle $|z| = 2$. Calculate the integral using residues.*

$$\int_C \frac{5z-4}{z(z-3)}\,dz$$

The function has singularities at $z = 0$ and $z = 3$, of which only $z = 0$ is inside C. Thus, the integral is given by the Residue Theorem

$$I = \int_C \frac{5z-4}{z(z-3)}\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{5z-4}{z(z-3)}$$

We expand the second factor to find

$$\begin{aligned}
\frac{5z-4}{z(z-3)} &= \left(\frac{4}{z} - 5\right)\cdot\frac{\frac{1}{3}}{1-\frac{z}{3}} \\
&= \left(\frac{4}{z} - 5\right)\cdot\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right) \\
&= \frac{4}{3z} + \left(\frac{4}{9} - \frac{5}{3}\right) + \left(\frac{4}{27} - \frac{5}{9}\right)z\cdots
\end{aligned}$$

so that

$$\operatorname*{Res}_{z=0} \frac{5z-4}{z(z-3)} = \frac{4}{3} \qquad \text{and} \qquad I = \frac{8\pi i}{3}.$$

13. *Let the degree of the polynomials*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \qquad (a_n \neq 0).$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m, \qquad (b_m \neq 0).$$

be such that $m \geq n+2$. By computing the residue at infinity, show that if all the zeros of $Q(z)$ are interior to a simple closed contour C, then

$$\int_C \frac{P(z)}{Q(z)}\,dz = 0.$$

See text problem 238[7]. Since C is outside all the zeros, then for R large enough, $P(z)/Q(z)$ is analytic in $R \le |z| < \infty$ so ∞ is an isolated singularity. Let C_0 be a negatively oriented circle C_0 (about 0) given by $|z| = R$. By the Residue Theorem at infinity

$$I = \int_C \frac{P(z)}{Q(z)}\,dz = -\int_{C_0} \frac{P(z)}{Q(z)}\,dz = -2\pi i \operatorname*{Res}_{z=\infty} \frac{P(z)}{Q(z)} = -2\pi i B.$$

Now, by the change of variables $\zeta = 1/z$ we get the formula

$$B = -\operatorname*{Res}_{\zeta=0} \frac{1}{\zeta^2} \frac{P\left(\frac{1}{\zeta}\right)}{Q\left(\frac{1}{\zeta}\right)}$$

Multiplying top and bottom by ζ^m we find

$$\begin{aligned}
\frac{1}{\zeta^2} \cdot \frac{P\left(\frac{1}{\zeta}\right)}{Q\left(\frac{1}{\zeta}\right)} &= \frac{1}{\zeta^2} \cdot \frac{a_0 + \dfrac{a_1}{\zeta} + \cdots + \dfrac{a_n}{\zeta^n}}{b_0 + \dfrac{b_1}{\zeta} + \cdots + \dfrac{b_m}{\zeta^m}} \\
&= \zeta^{m-n-2} \cdot \frac{a_n + a_{n-1}\zeta + \cdots + a_0\zeta^n}{b_m + b_{m-1}\zeta + \cdots + b_0\zeta^m} \\
&= \zeta^{m-n-2} g(\zeta).
\end{aligned}$$

Since $a_n \ne 0$ and $b_m \ne 0$ the rational function $g(\zeta)$ is analytic at $\zeta = 0$ such that $c_{m-n-2} = g(0) = a_n/b_m \ne 0$. Since $m - n - 2 \ge 0$, the function

$$\zeta^{m-n-2} g(\zeta) = c_{m-n-2}\zeta^{m-n-2} + c_{m-n-1}\zeta^{m-n-1} + \cdots$$

has no negative powers, thus the residue $B = 0$.

14. *Suppose that the function $f(z)$ is analytic throughout the entire plane except for a finite number of singular points z_1, z_2,…,z_n. Show that*

$$\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

See text problem 238[6]. Let C be a positively oriented (about 0) simple closed contour that encloses all of the singular points. Let C_0 be a circle $|z| = R$, where R is large enough to enclose C and the singular points, which is oriented negatively with respect to the origin. Then from the Residue Theorem for points inside,

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) \tag{1}$$

On the other hand, since f is analytic on C and C_0 and the annular region between, and by the Residue Theorem at infinity since $f(z)$ is analytic on $|z| \ge R$,

$$\frac{1}{2\pi i}\int_C f(z)\,dz = -\frac{1}{2\pi i}\int_{C_0} f(z)\,dz = -\operatorname*{Res}_{z=\infty} f(z) \tag{2}$$

Equating formulas (1) and (2) give the desired result.

15. *Show that the the point $z = \pi i$ is a pole. Determine the order of the pole m, and the corresponding residue B.*

$$f(z) = \frac{1 + \cosh z}{(z^2 + \pi^2)^3}$$

The denominator may be factored as

$$(z^2+\pi^2)^3 = (z-\pi i)^3(z+\pi i)^3$$

Using the fact that $\cosh(\pi i) = \cos(\pi) = -1$ and $\sinh(\pi i) = i\sin\pi = 0$, the numerator may be expressed as a power series by using the trigonometric identitiy

$$\begin{aligned}
1+\cosh z &= 1+\cosh\big(\pi i + (z-\pi i)\big)\\
&= 1+\cosh(\pi i)\cosh(z-\pi i) + \sinh(\pi i)\sinh(z-\pi i)\\
&= 1-\cosh(z-\pi i)\\
&= -\frac{1}{2}(z-\pi i)^2 - \frac{1}{24}(z-\pi i)^4 - \frac{1}{720}(z-\pi i)^6 - \cdots
\end{aligned}$$

so that we may rewrite

$$f(z) = \frac{1}{(z-\pi i)^3}\cdot\frac{1}{(z+\pi i)^3}\cdot(1+\cosh z) = \frac{\phi(z)}{z-\pi i}$$

where

$$\phi(z) = -\frac{1}{(z+\pi i)^3}\cdot\left(\frac{1}{2}+\frac{1}{24}(z-\pi i)^2+\frac{1}{720}(z-\pi i)^4+\cdots\right)$$

is analytic and non-vanishing at $z=\pi i$. Thus $z=\pi i$ is a pole of order $m=1$ and the the residue is the lowest coefficient, namely

$$B = \phi(\pi i) = \frac{1}{(\pi i+\pi i)^3}\cdot\lim_{z\to\pi i}\frac{1+\cosh z}{(z-\pi i)^2}$$

Using l'Hospital's Rule twice on these "$\frac{0}{0}$" forms one gets

$$B = \frac{i}{8\pi^3}\cdot\lim_{z\to\pi i}\frac{\sinh z}{2(z-\pi i)} = \frac{i}{8\pi^3}\cdot\lim_{z\to\pi i}\frac{\cosh z}{2} = -\frac{i}{16\pi^3}.$$

Equivalently, we could have expanded the first factor to find

$$\frac{1}{(z+\pi i)^3} = \left(\frac{\frac{1}{2\pi i}}{1+\frac{z-\pi i}{2\pi i}}\right)^3 = \left(\frac{1}{2\pi i}-\frac{z-\pi i}{(2\pi i)^2}+\cdots\right)^3 = \frac{i}{8\pi^3}-\frac{3}{16\pi^4}(z-\pi i)+\cdots$$

so that

$$\begin{aligned}
\phi(z) &= -\left(\frac{i}{8\pi^3}-\frac{3}{16\pi^4}(z-\pi i)+\cdots\right)\cdot\left(\frac{1}{2}+\frac{1}{24}(z-\pi i)^2+\frac{1}{720}(z-\pi i)^4+\cdots\right)\\
&= -\frac{i}{16\pi^3}+\frac{3}{32\pi^4}(z-\pi i)+\cdots
\end{aligned}$$

Thus confirming that

$$B = \phi(\pi i) = -\frac{i}{16\pi^3}.$$

16. Find the residue for each single valued branch at the singular point.

$$f(z) = \frac{\sqrt{z}}{z-1}$$

The square root function has two branches at $z=1$ given in $r>0$ and $-\pi<\theta<\pi$ for $n=0,1$ by

$$p(z) = z^{\frac{1}{2}} = \sqrt{r}\,e^{(\theta+2\pi i n)/2} = \pm\sqrt{r}e^{i\theta/2}$$

Near $z = 1$ both branches are analytic and $p(1) = \pm 1 \neq 0$. On the other hand, the denominator

$$q(z) = z - 1$$

has a simple zero at $z_0 = 1$. Thus the residue is given by

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{\pm\sqrt{1}}{1} = \pm 1.$$

17. *Let C_N be the positively oriented boundary of a square whose edges lie along the lines*

$$x = \pm\left(N + \frac{1}{2}\right) \qquad \textit{and} \qquad y = \pm\left(N + \frac{1}{2}\right)$$

where N is a positive integer. Show

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

Then, show that this integral tends to zero as $N \to \infty$ and conclude

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

See the text problem 254[6]. Observe that zero is a pole of order $m = 3$ for the function

$$f(z) = \frac{1}{z^2 \sin z} = \frac{1}{z^3\left(1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 - \cdots\right)}$$

Dividing, we find

$$\begin{array}{r|l}
 & 1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots \\
\hline
1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 - \cdots & 1 + 0 \cdot z^2 + 0z^4 + \cdots \\
 & \underline{1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 - \cdots} \\
 & \quad \frac{1}{6}z^2 - \frac{1}{120}z^4 + \cdots \\
 & \quad \underline{\frac{1}{6}z^2 - \frac{1}{36}z^4 + \cdots} \\
 & \qquad \frac{7}{360}z^4 + \cdots \\
 & \qquad \underline{\frac{7}{360}z^4 + \cdots} \\
 & \qquad 0 + \cdots
\end{array}$$

so that the second coefficient of $\phi(z)$ gives the

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{6}.$$

The other poles in the square are at $z = \pm n\pi$ where $n = 1, \ldots, N$. At $z_0 = \pm n\pi$, the function $f(z)$ has a simple pole. Writing $f(z) = p(z)/q(z)$ where $p(z) = 1$ and $q(z) = z^2 \sin z$, the residue

$$\operatorname*{Res}_{z=\pm n\pi} f(z) = \frac{p(\pm n\pi)}{q'(\pm n\pi)} = \frac{1}{2z_0 \sin z_0 + z_0^2 \cos z_0} = \frac{1}{0 + (\pm n\pi)^2 \cos(\pm n\pi)} = \frac{(-1)^n}{n^2\,\pi^2}$$

Hence, the Residue Theorem tells us the desired value of the integral

$$\frac{1}{2\pi i}\int_{C_N} f(z)\,dz = \sum_{n=-N}^{N} \operatorname*{Res}_{z=n\pi} f(z) = \frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\,\pi^2}$$

In my "Practice Problems for the Second Midterm," problem 17, I show that the integral tends to zero as $N \to \infty$. Thus, taking $N \to \infty$, we find

$$0 = \frac{1}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\,\pi^2}$$

Solving we find the desired sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

You may have seen another derivation of this awesome formula using Fourier Series.

18. *Evaluate the improper integral.*

$$\int_0^{\infty} \frac{x^2}{x^4+x^2+1}\,dx$$

[From Kenneth S. Miller, *Introduction to Advanced Complex Calculus,* Dover, New York, 1970, p. 135; republication of Harper & Brothers, 1960.] The integral is obtained as the limit

$$I = \lim_{R\to\infty} \int_0^R \frac{x^2}{x^4+x^2+1}\,dx$$

Because the integrand is bounded for $x \neq 0$ by an integrable function

$$\frac{x^2}{x^4+x^2+1} \le \frac{x^2}{x^2(x^2+1)} = \frac{1}{x^2+1}$$

which holds also for $x = 0$, the improper integral exists. Since the function

$$f(z) = \frac{z^2}{z^4+z^2+1}$$

is even, real on the real axis and the poles miss the real axis

$$I = \frac{1}{2}\lim_{R\to\infty} \int_{S_R} \frac{z^2}{z^4+z^2+1}\,dz$$

Where S_R is the line segment from $-R$ to R. Let C_R denote the semicircle $|z| = R$ with positive orientation in the upper half plane. Solving for the zeros of the denominator, z^2 satisfies a quadratic equation whose zeros are

$$z^2 = \frac{-1 \pm \sqrt{3}\,i}{2} = e^{\pm 2\pi i/3}$$

Thus the roots are $z = \pm e^{\pm\pi i/3}$ or

$$\alpha, \bar\alpha, \beta, \bar\beta \qquad \text{where} \qquad \alpha = \frac{1+\sqrt{3}\,i}{2}, \qquad \beta = \frac{-1+\sqrt{3}\,i}{2}$$

The simple closed contour $S_R + C_R$ encloses two roots α and β if $R > 1$. By the Residue Theorem

$$\int_{S_R+C_R} f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z)\right)$$

Let us estimate the integral over C_R. If $R > 1$ then the length of C_R is $L_R = \pi R$. The function is bounded for any $z \in C_R$ (for $R > 2$) by

$$\left|\frac{z^2}{z^4+z^2+1}\right| = \frac{|z|^2}{|z^4+z^2+1|} \le \frac{|z|^2}{\big||z^4| - |z^2+1|\big|} \le \frac{|z|^2}{|z|^4 - |z|^2 - 1} = \frac{R^2}{R^4 - R^2 - 1} = M_R$$

Thus the integral is bounded by

$$\left|\int_{C_R} \frac{z^2}{z^4+z^2+1}\,dz\right| \le L_R M_R = \frac{\pi R^3}{R^4 - R^2 - 1}$$

which tends to zero as $R \to \infty$. Computing the residues, using l'Hospital's Theorem,

$$\operatorname*{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha} \frac{(z-\alpha)z^2}{z^4+z^2+1} = \lim_{z\to\alpha} \frac{3z^2 - 2\alpha z}{4z^3+2z} = \frac{3\alpha^2 - 2\alpha^2}{\alpha(4\alpha^2+2)} = \frac{\alpha}{4\alpha^2+2} = \frac{\alpha}{2\sqrt{3}\,i}$$
$$\operatorname*{Res}_{z=\beta} f(z) = \lim_{z\to\beta} \frac{(z-\beta)z^2}{z^4+z^2+1} = \lim_{z\to\beta} \frac{3z^2 - 2\beta z}{4z^3+2z} = \frac{3\beta^2 - 2\beta^2}{\beta(4\beta^2+2)} = \frac{\beta}{4\beta^2+2} = \frac{\beta}{-2\sqrt{3}\,i}$$

where we have used $2\alpha^2 = -1 + \sqrt{3}\,i$ and $2\beta^2 = -1 - \sqrt{3}\,i$. Adding we find

$$\int_{S_R} f(z)\,dz + \int_{C_R} f(z)\,dz = \frac{2\pi\,i}{2\sqrt{3}\,i}(\alpha - \beta) = \frac{\pi}{\sqrt{3}}$$

because $\alpha - \beta = 1$. Letting $R \to \infty$ yields

$$I = \frac{1}{2}\int_{-\infty}^{\infty} f(z)\,dz = \frac{\pi}{2\sqrt{3}}.$$

19. *Use residues to evaluate the integral.*

$$I = \int_0^\infty \frac{x^3 \sin x\,dx}{(x^2+1)(x^2+4)}$$

Note that the function

$$f(z) = \frac{z^3}{(z^2+1)(z^2+4)} = \frac{z^3}{(z-i)(z+i)(z-2i)(z+2i)}$$

decays like $1/|z|$ and is not integrable by itself and needs the $\sin z$ factor to make it integrable. This integral may be integrated using Jordan's Lemma, because $f(z)$ is a rational function without poles on the real axis, analytic outside $|z| > 2$ and for any $|z| = R$,

$$|f(z)| \le \frac{R^3}{(R^2-1)(R^2-4)} = M_R$$

which tends to zero as $R \to \infty$. As usual, it is far easier to remember the trick in Jordan's Lemma that makes the estimates work, rather than the Lemma itself.

First of all, for z real, $f(z)\sin z = \Im\mathrm{m}(f(z)e^{iz})$ is an even function so that the improper integral exists if the limit exists,

$$I = \Im\mathrm{m}\left(\lim_{R\to\infty} \int_0^R f(z)e^{iz}\,dz\right) = \Im\mathrm{m}\left(\lim_{R\to\infty} \frac{1}{2}\int_{S_R} f(z)e^{iz}\,dz\right),$$

where S_R is the line segment from $-R$ to R. Let C_R be the semicircle $|z| = R$ in the upper halfplane in the counterclockwise direction. Estimating the integral,

$$\begin{aligned}
\left|\int_{C_R} f(z)e^{iz}\,dz\right| &= \left|\int_0^\pi f(Re^{i\theta})e^{i(R\cos\theta+iR\sin\theta)}\,Re^{i\theta}\,d\theta\right| \\
&\le \int_0^\pi |f(Re^{i\theta})|\,\left|e^{i(R\cos\theta+iR\sin\theta)}\right|\,R\,d\theta \\
&\le \int_0^\pi M_R\,e^{-R\sin\theta}\,R\,d\theta \\
&= \frac{R^4}{(R^2-1)(R^2-4)}\int_0^\pi e^{-R\sin\theta}\,d\theta \\
&= \frac{2R^4}{(R^2-1)(R^2-4)}\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta
\end{aligned}$$

where we use the fact that $\sin\theta$ is symmetric to the left and right of $\theta = \pi/2$. Now Jordan's trick is the inequality for $0 \le \theta \le \pi/2$

$$\sin\theta \ge \frac{2\theta}{\pi}.$$

which just says that the straight line from the origin to $(\pi/2, 1)$ lies under the curve $y = \sin\theta$. Using it in the integral

$$\begin{aligned}
\left|\int_{C_R} f(z)e^{iz}\,dz\right| &\le \frac{2R^4}{(R^2-1)(R^2-4)}\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta \\
&= \frac{\pi R^3}{(R^2-1)(R^2-4)}\left(1-e^{-R}\right)
\end{aligned}$$

which tends to zero as $R \to \infty$. The function $f(z)e^{iz}$ has simple poles inside the contour at i and $2i$. Thus the Residue Theorem tells us that

$$\int_{S_R+C_R} f(z)e^{iz}\,dz = 2\pi i\left(\operatorname*{Res}_{\mathrm{z=i}} f(z)e^{iz} + \operatorname*{Res}_{\mathrm{z=2i}} f(z)e^{iz}\right)$$

The residues may be evaluated as $\phi(z_0)$ which is

$$\begin{aligned}
\operatorname*{Res}_{\mathrm{z=i}} f(z)e^{iz} &= \phi(i) = \frac{i^3e^{i\cdot i}}{(i+i)(i-2i)(i+2i)} = -\frac{e^{-1}}{6} \\
\operatorname*{Res}_{\mathrm{z=2i}} f(z)e^{iz} &= \phi(2i) = \frac{(2i)^3e^{i\cdot 2i}}{(2i-i)(2i+i)(2i+2i)} = \frac{2e^{-2}}{3}
\end{aligned}$$

Letting $R \to \infty$, we see that the integral

$$I = \Im\mathrm{m}\left[\pi i\left(-\frac{e^{-1}}{6} + \frac{2e^{-2}}{3}\right)\right] = \frac{\pi}{6e^2}(4-e).$$

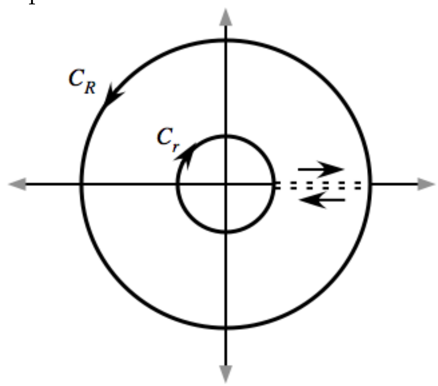
20. *Derive the integration formula*

$$I = \int_0^\infty \frac{\sqrt[3]{x}}{(x+8)(x+1)}\,dx = \frac{2\pi}{7\sqrt{3}}$$

See the text problem 283[4]. Using the branch $r > 0$ and $0 \le \theta \le 2\pi$ so that

$$f(z) = \frac{z^{1/3}}{(z+8)(z+1)} = \frac{e^{(1/3)\log z}}{(z+8)(z+1)}$$

$f(z)$ is not analytic at zero and there are two branches on the positive real axis for $\theta = 0$ and $\theta = 2\pi$. We shall take the contour that consists of L_1 from r to R where $0 < r < R$ and $\theta = 0$, C^R which is the circle $|z| = R$ from $\theta = 0$ to $\theta = 2\pi$, L_2 line segment from R to r at $\theta = 2\pi$, and C_r which is the circle $|z| = r$ from $\theta = 2\pi$ to $\theta = 0$. The two segments L_1 and L_2 overlap but correspond to different branches of the cube root.



The improper integral

$$I = \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{\sqrt[3]{x}}{(x+8)(x+1)}\,dx = \lim_{\substack{r\to 0\\ R\to\infty}} \int_{L_1} \frac{e^{(1/3)\log z}}{(z+8)(z+1)}\,dz$$

since $\theta = 0$ on L_1. On L_2, $\theta = 2\pi$ so that

$$e^{(1/3)\log z} = e^{(1/3)(\ln x + 2\pi i)} = e^{2\pi i/3} e^{(1/3)\ln x}$$

so that, taking the revered direction into account,

$$\int_{L_2} \frac{e^{(1/3)\log z}}{(z+8)(z+1)}\,dz = -e^{2\pi i/3} \int_{L_1} \frac{e^{(1/3)\log z}}{(z+8)(z+1)}\,dz$$

Finally we check that the other integrals vanish. Indeed for $|z| = R > 8$, because $|z+1| \geq ||z| - |1|| \geq |z| - 1 = R - 1$ and $|z+8| \geq R - 8$,

$$\left|\frac{e^{(1/3)(\ln R + \theta i)}}{(z+8)(z+1)}\right| \leq \frac{R^{1/3}}{(R-8)(R-1)} = M_R$$

Since C_R has length $L_R = 2\pi R$ we get the estimate

$$\left|\int_{C_R} \frac{e^{(1/3)(\ln R + \theta i)}}{(z+8)(z+1)}\,dz\right| \leq M_R L_R = \frac{2\pi R^{4/3}}{(R-8)(R-1)}$$

which tends to zero as $R \to \infty$. For $0 < |z| = r < 1$, because $|z+1| \geq |1-|z|| \geq 1-|z| = 1-r$ and $|z+8| \geq 8 - r$,

$$\left|\frac{e^{(1/3)(\ln r + \theta i)}}{(z+8)(z+1)}\right| \leq \frac{R^{1/3}}{(8-r)(1-r)} = M_R$$

Since C_r has length $L_r = 2\pi r$ we get the estimate

$$\left|\int_{C_r} \frac{e^{(1/3)(\ln R + \theta i)}}{(z+8)(z+1)}\,dz\right| \leq M_r L_r = \frac{2\pi r^{4/3}}{(8-r)(1-r)}$$

which tends to zero as $r \to 0$.

Now the contour $L_1 + C_R + L_2 + C_r$ encloses the poles $z = -1$ and $z = -8$. By the Residue Theorem,

$$\int_{L_1+C_R+L_2+C_r} \frac{z^{1/3}}{(z+8)(z+1)}\,dz = 2\pi i \left(\operatorname*{Res}_{\text{z=-1}} f(z) + \operatorname*{Res}_{\text{z=-8}} f(z)\right) \tag{3}$$

Computing, we find

$$\operatorname*{Res}_{\text{z=-1}} f(z) = \phi(-1) = \frac{e^{(1/3)(\ln 1+\pi i)}}{(-1+8)} = \frac{e^{\pi i/3}}{7}$$

$$\operatorname*{Res}_{\text{z=-8}} f(z) = \phi(-8) = \frac{e^{(1/3)(\ln 8+\pi i)}}{(-8+1)} = -\frac{2e^{\pi i/3}}{7}$$

Taking the limit as $r \to 0$ and $R \to \infty$ in (3),

$$(1 - e^{2\pi i/3})I = 2\pi i \left(\frac{e^{\pi i/3}}{7} - \frac{2e^{\pi i/3}}{7}\right).$$

It follows that

$$-\frac{\sqrt{3}}{2} I = -\sin\left(\frac{\pi}{3}\right) I = \frac{e^{-\pi i/3} - e^{\pi i/3}}{2i} I = -\frac{\pi}{7}$$

so

$$I = \frac{2\pi}{7\sqrt{3}}.$$