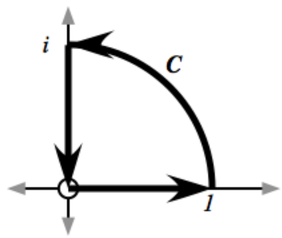
Math 3160 § 1. **Second Midterm Exam** Name: Exam Solutions
Treibergs $\sigma\tau$ March 26, 2015

1. *Let C be the contour consisting of the quarter circle $|z| = 1$ in the first quadrant and the line segments from i to 0 to 1 with counterclockwise orientation as in the figure. Find*

$$I = \int_C |z|\,dz$$



The function $|z|$ is not analytic on the region bounded by C so we must perform the parameterized integrals or replace parts of the integral by equivalent expressions.

If we parameterize each part, let's denote the line segment from 0 to 1 by C_1, the arc from 1 to i by C_2 and the segment from i to 0 by C_3. A parameterization of C_1 is given by $z_1(t) = t$, $0 \le t \le 1$ with $|z_1(t)| = t$ and $z_1'(t) = 1$. A parameterization of C_2 is given by $z_2(t) = e^{it}$, $0 \le t \le \pi/2$ with $|z_2(t)| = 1$ and $z_2'(t) = ie^{it}$. A parameterization of C_3 is given by $z_3(t) = -it$, $-1 \le t \le 0$ and $|z_3(t)| = -t$ and $z_3'(t) = -i$. Thus

$$\begin{aligned} I &= \int_{C_1} |z|\,dz + \int_{C_2} |z|\,dz + \int_{C_3} |z|\,dz \\ &= \int_0^1 t\,dt + \int_0^{\pi/2} ie^{it}\,dt + \int_{-1}^0 (-t)(-i)\,dt \\ &= \left[\frac{t^2}{2}\right]_0^1 + \left[e^{it}\right]_0^{\pi/2} + \left[\frac{it^2}{2}\right]_{-1}^0 \\ &= \left(\frac12 - 0\right) + (i-1) + \left(0 - \frac{i}{2}\right) = \frac12(i-1). \end{aligned}$$

Alternately, we may replace each integrand by an analytic function. On C_1, $z = x$ so we may replace $|z| = x$ by z. On C_2, $|z| = 1$. On C_3, $z = iy$ so we may replace $|z| = y$ by $-iz$. The three integrals are then evaluated using antiderivatives

$$\begin{aligned} I &= \int_{C_1} |z|\,dz + \int_{C_2} |z|\,dz + \int_{C_3} |z|\,dz \\ &= \int_{C_1} z\,dz + \int_{C_2} dz + \int_{C_3} -iz\,dz \\ &= \left[\frac{z^2}{2}\right]_0^1 + \left[z\right]_1^i + \left[-\frac{iz^2}{2}\right]_i^0 \\ &= \left(\frac12 - 0\right) + (i-1) + \left(-0 - \frac{i}{2}\right) = \frac12(i-1). \end{aligned}$$

2. (a) *Find all roots of the equation* $\sinh z = 1.$

One solution is to solve real equations for the real and imaginary parts. Thus

$$\sinh(x+iy) = \sinh x\, \cosh iy \,+\, \cosh x\, \sinh iy = \sinh x\, \cos y \,+\, i \cosh x\, \sin y = 1$$

yields the two real equations

$$\begin{aligned} \sinh x \cos y &= 1, \\ \cosh x \sin y &= 0. \end{aligned}$$

Because $\cosh x \geq 1$ for all x, the second equation reduces to

$$\sin y = 0$$

which has the solution $y = \pi n$ for $n \in \mathbf{Z}$. Then the first equation becomes

$$1 = \cos y\, \sinh x = \cos n\pi\, \sinh x = (-1)^n \sinh x.$$

Since $\sinh x$ is an odd function, the roots are $x = (-1)^n \sinh^{-1} 1 = (-1)^n w$. To see what this is we solve

$$1 = \sinh w = \frac{e^w - e^{-w}}{2}$$

or

$$0 = e^{2w} - 2e^w - 1.$$

By the quadratic formula,

$$e^w = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-1)}}{2} = 1 \pm \sqrt{2}$$

Since the real exponential is positive, only th "+" root works, so $w = \ln(1+\sqrt{2})$. Putting together the solution,

$$z = (-1)^n \ln(1+\sqrt{2}) + n\pi i. \qquad \text{for } n \in \mathbf{Z}.$$

Another solution is to use complex logs and square roots. Replacing the hyperbolic sine with its definition,

$$1 = \sinh z = \frac{e^z - e^{-z}}{2}$$

so

$$e^{2z} - 2e^z - 1 = 0.$$

Using the quadratic formula

$$e^z = \frac{2 + \left((-2)^2 + 4\right)^{1/2}}{2} = 1 \pm \sqrt{2}$$

where the double valued square root function is used. Then taking logarithms

$$z = \log\left(1 \pm \sqrt{2}\right)$$

For the "$+\sqrt{2}$" roots, we get

$$\log\left(1+\sqrt{2}\right) = \ln\left(1+\sqrt{2}\right) + 2n\pi i; \qquad \text{for } n \in \mathbf{Z}.$$

For the "$-\sqrt{2}$" roots, because $1-\sqrt{2}<0$ we get

$$\log\left(1-\sqrt{2}\right) = \ln\left(\sqrt{2}-1\right) + (2n+1)\pi i; \qquad \text{for } n \in \mathbf{Z}.$$

This suffices, but can be sharpened a bit. Noting that

$$\sqrt{2}-1 = \frac{1}{1+\sqrt{2}}$$

we may write the "$-\sqrt{2}$" branch roots as

$$\log\left(1-\sqrt{2}\right) = -\ln\left(1+\sqrt{2}\right) + (2n+1)\pi i; \qquad \text{for } n \in \mathbf{Z}.$$

Combining the two branches yields

$$z = (-1)^n \ln(1+\sqrt{2}) + n\pi i; \qquad \text{for } n \in \mathbf{Z}.$$

(b) *Assuming that* $|\Re\mathrm{e}\, z| \le a$ *for some* $a > 0$, *find a bound for* $|\sinh z|$.

Expanding the hyperbolic sine we find

$$\sinh(x+iy) = \sinh x \cosh(iy) + \cosh x \sinh(iy) = \sinh x \cos y + i \cosh x \sin y$$

so

$$\begin{aligned} |\sinh z|^2 &= \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y \\ &= \sinh^2 x \cos^2 y + (1+\sinh^2 x)\sin^2 y \\ &= \sinh^2 x(\cos^2 y + \sin^2 y) + \sin^2 y \\ &= \sinh^2 x + \sin^2 y. \end{aligned}$$

Thus if $|x| = |\Re\mathrm{e}\, z| \le a$ then since $\sinh x$ is odd and increasing,

$$|\sinh x| = \sinh |x| \le \sinh a$$

so using also $\sin^2 y \le 1$,

$$|\sinh z|^2 \le \sinh^2 a + 1 = \cosh^2 a.$$

or

$$|\sinh z| \le \cosh a.$$

3. Quick Answers.

(a) *State why the function* $\quad h(x,y) = \ln(x^2+y^2) \quad$ *is harmonic in* $\Re\mathrm{e}\, z > 0$.

$h(x,y)$ is the real part of the function $2\,\mathrm{Log}\, z = 2\ln r + 2\theta i = \ln(x^2+y^2) + 2\theta i$ which is analytic in the region $r > 0$ and $-\pi < \theta < \pi$, which includes the right half plane.

(b) *Suppose f is analytic in the halfplane $\{z \in \mathbf{C} : \Re e\, z > -1\}$, g is analytic to the halfplane $\{z \in \mathbf{C} : \Re e\, z < 1\}$ and they are equal $f(z) = g(z)$ if z is a point of the imaginary axis $\Re e\, z = 0$. Explain why $h(z)$ defines an entire function, where*

$$h(z) = \begin{cases} f(z), & \text{if } \Re e\, z > 0; \\ g(z), & \text{if } \Re e\, z \le 0. \end{cases}$$

Since f and g are analytic in their own domains and agree on any line segment in the imaginary axis common to both domains, they must be continuations of each other by identity principle (also called the coincidence principle). Since the union of the two half spaces is the entire plane, the resulting function is an analytic function on the whole plane, thus is entire.

(c) *Give an example of a domain $\mathcal{D} \subset \mathbf{C}$, a single valued analytic function $f : \mathcal{D} \to \mathbf{C}$, two points $z_1, z_2 \in \mathcal{D}$ and two contours C_1 and C_2, both starting at z_1 and ending at z_2, such that*

$$\int_{C_1} f(z)\, dz \ne \int_{C_2} f(z)\, dz.$$

Let $\mathcal{D} = \mathbf{C} - \{0\}$ be the punctured plane, $f(z) = \dfrac{1}{z}$, $z_1 = 1$, $z_2 = -1$, C_1 be the arc $z(\theta) = e^{i\theta}$ for $0 \le \theta \le \pi$ and C_2 be the arc $z(\theta) = e^{-i\theta}$ for $0 \le \theta \le \pi$. Then

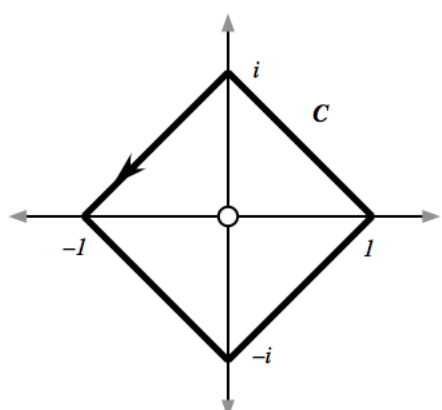
$$\int_{C_1} f(z)\, dz = \int_0^\pi \frac{ie^{i\theta}\, d\theta}{e^{i\theta}} = \int_0^\pi i\, d\theta = \pi i$$

whereas

$$\int_{C_2} f(z)\, dz = \int_0^\pi \frac{-ie^{-i\theta}\, d\theta}{e^{-i\theta}} = -\int_0^\pi i\, d\theta = -\pi i.$$

4. *Let C be the diamond-shaped closed contour consisting of line segments from 1 to i to -1 to $-i$ and back to 1 as in the figure. Assuming that the principal fifth root is used, find*

$$J = \int_C z^{1/5}\, dz$$



The principal value is $f(z) = z^{1/5} = \sqrt[5]{r}\, e^{i\theta/5}$ for $r > 0$ and $-\pi < \theta \le \pi$ which is discontinuous at $\theta = \pi$. One can write parameterized integrals for $f(z)\, dz$ and notice that the integration over an open interval extends continuously to the closed interval and integrate as usual with real calculus. We prefer to use antiderivatives. Split the contour into $C = C_1 + C_2$ where C_1 is the upper half from 1 to i to -1. On C_1, the principal logarithm agrees with $\log z = \ln r + i\theta$ in the branch $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Using this logarithm, the antiderivative is

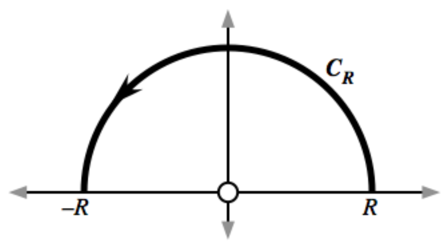
$$F(z) = \frac{5}{6} r \sqrt[5]{r}\, e^{6i\theta/5}$$

On C_2, the principal logarithm agrees with $\log z = \ln r + i\theta$ in the branch $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$. The antiderivative is again $F(z)$, but using this logarithm instead. Thus we may compute

$$\begin{aligned} J &= \int_{C_1} z^{1/5}\, dz + \int_{C_2} z^{1/5}\, dz = \int_1^{-1} z^{1/5}\, dz + \int_{-1}^{1} z^{1/5}\, dz = \Big[F(z)\Big]_1^{-1} + \Big[F(z)\Big]_{-1}^{1} \\ &= \frac{5}{6}\left[e^{6\pi i/5} - e^{0i}\right] + \frac{5}{6}\left[e^{0i} - e^{-6\pi i/5}\right] = \frac{5}{6}\left(e^{6\pi i/5} - e^{-6\pi i/5}\right) \\ &= \frac{5}{3}\sinh\left(\frac{6\pi i}{5}\right) = \frac{5i}{3}\sin\left(\frac{6\pi}{5}\right) \end{aligned}$$

5. *Let C_R be the upper half of the circle $|z| = R$ from R to $-R$ as in the figure. Show that the integral over C_R tends to zero as $R \to \infty$.*

$$K_R = \int_{C_R} \frac{e^{iz}\, dz}{z^2 + z + 2}$$



Observe that the length of the semicircular contour is $L_R = \pi R$. Also C_R is parameterized $z(\theta) = Re^{i\theta} = R\cos\theta + iR\sin\theta$ thus

$$|e^{iz}| = |e^{iR(\cos\theta + i\sin\theta)}| = |e^{-R\sin\theta} e^{iR\cos\theta}| = e^{-R\sin\theta}.$$

Since $0 \le \theta \le \pi$ we have $0 \le \sin\theta \le 1$ so that in this range $-R\sin\theta \le 0$ so

$$|e^{iz}| \le e^0 = 1.$$

It follows that for $z \in C_R$ and $R > 2$, by the reverse triangle inequality $|z^2 + z + 2| \ge \big||z|^2 - |z+2|\big| \ge |z|^2 - |z+2| \ge |z|^2 - |z| - 2$, the integrand is bounded by

$$\left|\frac{e^{iz}}{z^2 + z + 2}\right| = \frac{|e^{iz}|}{|z^2 + z + 2|} \le \frac{1}{|z|^2 - |z| - 2} = \frac{1}{R^2 - R - 2} = M_R$$

Then the contour integral has the bound

$$|K_R| = \left|\int_{C_R} \frac{e^{iz}\, dz}{z^2 + z + 2}\right| \le L_R M_R = \frac{\pi R}{R^2 - R - 2}$$

which tends to zero as $R \to \infty$.