Math 3160 § 1. **First Midterm Exam** Name: Solutions
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1. *Let $z = -1 + \sqrt{3}\, i$. Write $z$ in polar coordinates. Find $|z|$ and the principal argument* $\operatorname{Arg} z$*. Find $z^7$ in polar and rectangular coordinates. Find $\sqrt[4]{z}$ and $z^{\frac{1}{4}}$.*

$$r = |z| = \sqrt{x^2+y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

and $\Theta = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$. Since $-\pi < \Theta \le \pi$, $\Theta$ is the principal argument, thus $\operatorname{Arg} z = \dfrac{2\pi}{3}$ and $z = 2\exp\left(\dfrac{2\pi}{3} i\right)$.

$$\begin{aligned} z^7 = (re^{i\Theta})^7 = 2^7 \exp\left(7 \cdot \frac{2\pi}{3} i\right) = 128\exp\left(\frac{14\pi}{3} i\right) = 128 \exp\left(4\pi i + \frac{2\pi}{3} i\right) \\ = 128\exp\left(\frac{2\pi}{3} i\right) = 64 \cdot 2\exp\left(\frac{2\pi}{3} i\right) = 64(-1+\sqrt{3}\, i) = -64 + 64\sqrt{3} i. \end{aligned}$$

The principal fourth root is

$$\sqrt[4]{z} = (re^{i\Theta})^{\frac{1}{4}} = \sqrt[4]{2}\exp\left(\frac{1}{4}\cdot\frac{2\pi}{3} i\right) = \sqrt[4]{2}\exp\left(\frac{\pi}{6} i\right) = \sqrt[4]{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

The set of fourth roots is

$$\begin{aligned} z^{\frac{1}{4}} &= \left\{\sqrt[4]{2}\exp\left(\frac{1}{4}\cdot\left(\frac{2\pi}{3} + 2\pi k\right) i\right) : k \in \mathbf{Z}\right\} \\ &= \left\{\sqrt[4]{2}\exp\left(\frac{\pi i}{6} + \frac{\pi i k}{2}\right) : k = 0,1,2,3\right\} \\ &= \left\{\sqrt[4]{2}\exp\left(\frac{\pi i}{6}\right)\omega^k : k = 0,1,2,3\right\} \\ &= \left\{\sqrt[4]{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right), \sqrt[4]{2}\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right), -\sqrt[4]{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right), \sqrt[4]{2}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\right\} \end{aligned}$$

where $\omega = i$ is a primitive fourth root of unity.

2. *Determine for which $z$ the complex derivative $f'(z)$ exists. Find $f'(z)$ at those points.*

$$f(z) = e^{-2y}\sin 2x - ie^{-2y}\cos 2x.$$

The real and imaginary parts of $f$ and their partial derivatives are

$$\begin{aligned} u(x,y) &= e^{-2y}\sin 2x, & v(x,y) &= -e^{-2y}\cos 2x \\ u_x(x,y) &= 2e^{-2y}\cos 2x, & v_x(x,y) &= 2e^{-2y}\sin 2x \\ u_y(x,y) &= -2e^{-2y}\sin 2x, & v_y(x,y) &= 2e^{-2y}\cos 2x \end{aligned}$$

The four partial derivative functions (1) exist at all points of $\mathbf{C}$ since $f$ is a product and sum of differentiable functions, (2) are continuous at all points of $\mathbf{C}$ since the $u_x$, $u_y$, $v_x$, $v_y$ are product and sum of continuous functions and (3) the Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold at all points of $\mathbf{C}$. Thus by the theorem giving sufficient conditions for differentialility, $f$ is differentiable at all points $z \in \mathbf{C}$. The value of the derivative is given by the formula

$$f'(z) = u_x(x,y) + iv_x(x,y) = 2e^{-2y}\cos 2x + 2ie^{-2y}\sin 2x$$

3. (a) *Show* $\lim\limits_{z\to\infty} \dfrac{1+2z+3z^2}{4+5z+6z^2} = \dfrac{1}{2}$.

The infinite limit exists if the finite limit exists and has the same value. Change variables by $z = \frac{1}{\zeta}$ we find

$$\lim_{z\to\infty} \frac{1+2z+3z^2}{4+5z+6z^2} = \lim_{\zeta\to 0} \frac{1+\frac{2}{\zeta}+\frac{3}{\zeta^2}}{4+\frac{5}{\zeta}+\frac{6}{\zeta^2}} = \lim_{\zeta\to 0} \frac{\zeta^2+2\zeta+3}{4\zeta^2+5\zeta+6} = \frac{0+0+3}{0+0+6} = \frac{1}{2}.$$

(b) *Find the set in the $z$-plane whose image under the transformation $w = z^2$ is the fourth quadrant $\{w \in \mathbf{C} : \Re\mathrm{e}\, w > 0$ and $\Im\mathrm{m}\, w < 0\}$.*

One solution is to observe that $w = \rho e^{i\phi}$ where $\rho > 0$ and $-\dfrac{\pi}{2} < \phi < 0$. Since $w = z^2 = (re^{i\theta})^2 = r^2 e^{2i\theta} = r^2 e^{2i\theta - 2\pi i}$ we want $r = \sqrt{\rho}$ and either $\phi = 2\theta$ or $\phi = 2\theta - 2\pi$. But this is the set

$$\left\{ re^{i\theta} : \ r > 0 \text{ and either } -\frac{\pi}{4} < \theta < 0 \text{ or } \frac{3\pi}{4} < \theta < \pi \right\}$$

which is the union of two sectors.

Another solution is to solve the inequalities

$$\begin{aligned} 0 &< \Re\mathrm{e}\, w = x^2 - y^2, \\ 0 &> \Im\mathrm{m}\, w = 2xy. \end{aligned}$$

The solution is in the second and fourth quadrants where $0 < y < -x$ or $-x < y < 0$, respectively.

(c) *Let $E = \{z \in \mathbf{C} : |z - 2i| > 1\}$. Answer the following giving short reasons. Is the set $E$ is bounded? Is $E$ connected? Is $E$ open, closed or neither open nor closed? Does $E$ have any accumulation points? (If it has accumulation points, find one.)*

$E$ is the region strictly outside a circle of radius one and center $2i$. $E$ is unbounded since it is the exterior of a disk and is not contained in any finite ball.

$E$ is connected since between any pair of points of $E$ there can be drawn a piecewise linear path from point to point staying within $E$.

$E$ is open because around every point of $E$ there is open disk about the point strictly inside $E$.

$E$ has accumulation points. In fact, every point of $z \in E$ is an accumulation point. Since $E$ is open, there is a disk $D \subset E$ with center $z$. Every punctured disk about $z$ meets points of $E$, namely those in $D$. The points on the bounding circle are also accumulation points

4. (a) *Prove that the following limit does not exist:* $\lim\limits_{z\to 0} \dfrac{\Im\mathrm{m}\, z}{\bar{z}}$

Let $f(z) = \dfrac{\Im\mathrm{m}\, z}{\bar{z}}$. The limiting value must be consistent for different approaches to 0, but it is not here. Taking the approach along the $x$-axis $z = x + 0i$, $f(x+0i) = \dfrac{0}{x} = 0$ so $\lim\limits_{(x,0)\to(0,0)} f(x+0i) = 0$. Taking the approach along the $y$-axis, $z = 0 + yi$ gives $f(0+yi) = \dfrac{y}{-yi} = i$ so $\lim\limits_{(0,y)\to(0,0)} f(0+yi) = i$. Hence there is no complex limit.

(b) *Let $f : \mathbf{C} \to \mathbf{C}$ be a function and $z_0 \in \mathbf{C}$. State the DEFINITION: The function is complex differentiable at $z_0$. Using the rules for differentiation, find $f'(z_0)$ when $f(z) = \dfrac{z}{1-z}$ and $z_0 \neq 1$. Then using the definition instead, give a direct proof that $f(z) = \dfrac{z}{1-z}$ is complex differentiable at $z_0 = 0$.*

We say that the function $f : \mathbf{C} \to \mathbf{C}$ is differentiable at $z_0 \in \mathbf{C}$ if the following limit exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

The value of the limit is the complex derivative $f'(z_0)$.

The quotient rule for differentiation yields

$$f'(z) = \frac{1 \cdot (1-z) - z \cdot (-1)}{(1-z)^2} = \frac{1}{(1-z)^2}.$$

To show that the derivative exists, we need to find the limit of the difference quotient at $z_0 = 0$ where $f(z_0) = 0$:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to 0} \frac{\frac{z}{1-z} - 0}{z - 0} = \lim_{z \to 0} \frac{1}{1-z} = \frac{1}{1-0} = 1.$$

Since we can compute a limit of the difference quotients, the function is differentiable at $z_0$.

5. *Let $f : \mathbf{C} \to \mathbf{C}$ be a function and $z_0, w_0 \in \mathbf{C}$ be points. State the definition*

$$w_0 = \lim_{z \to z_0} f(z).$$

*Using the definition of the limit, prove that*

$$\lim_{z \to i} (3z + 4i\,\overline{z}) = 4 + 3i$$

The statement that $f(z)$ has a limit $w_0$ as $z$ approaches $z_0$ means that for every positive number $\varepsilon$ there is a positive number $\delta$ such that

$$|f(z) - w_0| < \varepsilon \qquad \text{whenever } 0 < |z - z_0| < \delta. \tag{1}$$

To show that

$$\lim_{z \to i} (3z + 4i\,\overline{z}) = 4 + 3i$$

observe that by the triangle inequality,

$$\begin{aligned} |f(z) - w_0| &= |3z + 4i\,\overline{z} - 4 - 3i| \\ &= |3(z-i) + 3i + 4i\,\overline{(z-i)} - 4i\overline{(-i)} - 4 - 3i| \\ &= |3(z-i) + 4i\,\overline{(z-i)}| \\ &\le 3|z-i| + 4|\overline{(z-i)}| = 3|z-i| + 4|z-i| = 7|z-i|. \end{aligned}$$

Now for any positive number $\varepsilon$, let $\delta = \dfrac{\varepsilon}{7}$. For any $z \in \mathbf{C}$ such that $0 < |z - z_0| < \delta$ we have using the observation that

$$|f(z) - w_0| = |3z + 4i\,\overline{z} - 4 - 3i| \le 7|z - i| < 7\delta = \epsilon.$$

Thus we have satisfied the condition (1) so the limit exists and equals $w_0 = 4 + 3i$.