Math 3160 § 1. **Second Midterm Exam** Name: Practice Problems
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1. *Determine the singular points of the function and state why the function is analytic everywhere else:*
$$f(z) = \frac{z^3 - 1}{(z+1)(z^2 + 3z + 7)}$$

Points z_0 where the function fails to be analytic, but is analytic at some point in every neighborhood of z_0 are called *singular points.* The function is *analytic* at a point z_0 if it is analytic in some neighborhood, an open set containing z_0.

The given function blows up where the denominator is zero (and the numerator is not zero.) For this function, this occurs at the points

$$z_0 = -1 \qquad \text{and} \qquad z_0 = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 7}}{2} = -\frac{3}{2} \pm \frac{\sqrt{19}}{2} i.$$

Away from these isolated points the denominator is nonzero so the complex derivative exists because it is a rational function of z. Thus every other point $\zeta \in \mathbf{C}$ is some distance $r > 0$ from these points, thus is analytic because the function is differentiable in the open disk of radius r about ζ. Also these points are singular because the there is no derivative at these points and every open set containing them contains other points where the function is analytic.

2. *Let A be an open subset of $\mathbf{C}$ and $A^* = \{z \in \mathbf{C} : \bar{z} \in A\}$ be its reflection across the real axis. Suppose that $f(z)$ is analytic on A. Show that $\overline{f(\bar{z})}$ is analytic on A^*.*

First observe that A^* is open. That is because every point $Z \in A^*$ has an open neighborhood $U = B^*$, namely, the reflection of the little open disk about $z \in B \subset A$.

Second, if $f(z) = u(x, y) + iv(x, y)$, then $g(z) = \overline{f(\bar{z})} = u(x, -y) - iv(x, -y)$ and we show that the Cauchy Riemann equations hold for g. By the Cauchy Riemann Equations for f,

$$\frac{\partial}{\partial x}(\Re\mathrm{e}\, g) = \frac{\partial}{\partial x} u(x, -y) = \frac{\partial}{\partial x} u \bigg|_{(x,-y)} = \frac{\partial}{\partial y} v \bigg|_{(x,-y)} = \frac{\partial}{\partial y}[-v(x, -y)] = \frac{\partial}{\partial y}(\Im\mathrm{m}\, g)$$

and

$$\frac{\partial}{\partial y}(\Re\mathrm{e}\, g) = \frac{\partial}{\partial y} u(x, -y) = -\frac{\partial}{\partial y} u \bigg|_{(x,-y)} = \frac{\partial}{\partial x} v \bigg|_{(x,-y)} = -\frac{\partial}{\partial x}[-v(x, -y)] = -\frac{\partial}{\partial x}(\Im\mathrm{m}\, g).$$

For every point of A^*, both real partial derivatives exist and are continuous for g because they are for f on A. Since the Cauchy Riemann equations hold as well, g is analytic on A^*.

3. *Suppose that $f(z) = u(x, y) + iv(x, y)$ is an analytic function defined on a domain $D \subset \mathbf{C}$. If $au(x, y) + bv(x, y) = c$, where a, b, c are real constants not all zero, then $f(z)$ is constant on A. Is the result still valid if a, b, c were allowed to be complex constants?*

Observe that we can't have both $a = b = 0$ because that implies that $c = 0$ too. Differentiating we find
$$\begin{aligned} 0 &= au_x + bv_x \\ 0 &= au_y + bv_y = bu_x - av_x. \end{aligned} \qquad \begin{vmatrix} a & b \\ b & -a \end{vmatrix} = -a^2 - b^2$$

Since the determinant of the coefficient matrix is strictly negative if a and b are real and not both zero, we must have $u_x = v_x = 0$ everywhere in D. Hence $f'(z) = u_x + iv_x = 0$ and since D is connected, f is constant.

The result continues to hold if $a, b.c$ are complex. If $a^2 + b^2 \neq 0$ then the same argument shows $f' = 0$ on D. But if $b^2 = -a^2$ then $b = \pm ia$, both nonzero, and the system has only one independent equation,

$$0 = au_x \pm aiv_x = a(u_x \pm iv_x) \qquad \Longrightarrow \qquad 0 = u_x \pm iv_x.$$

But since both u_x and v_x are real, the real and imaginary parts of this equation say $u_x = v_x = 0$ so f is again constant on D.

4. *Let f be an entire function that equals a polynomial on $[0,1]$ on the real exis. Show that f is a polynomial.*

   Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ on $[0,1]$. Then $f(z)$ and $a_0 + a_1 z + \cdots + a_n z^n$ agree on $[0,1]$, and both are analytic on $\mathbf{C}$ (that is, both are entire). By the Identity Theorem, if two analytic functions agree on a line segment contained in the domain of both, then they agree over the whole domain.

5. *Prove the identity:* $\quad \sinh e^z + \sinh e^{-z} = 2\sinh(\cosh z)\cosh(\sinh z)$

   Identities are proved starting from one side of the equality and deducing the other side.

$$\begin{aligned}
\sinh e^z + \sinh e^{-z} &= \frac{\exp(e^z) - \exp(-e^z)}{2} + \frac{\exp(e^{-z}) - \exp(-e^{-z})}{2} \\
&= \frac{1}{2}\Bigg\{\exp\left(\frac{e^z + e^{-z}}{2} + \frac{e^z - e^{-z}}{2}\right) - \exp\left(-\frac{e^z + e^{-z}}{2} - \frac{e^z - e^{-z}}{2}\right) \\
&\qquad + \exp\left(\frac{e^z + e^{-z}}{2} - \frac{e^z - e^{-z}}{2}\right) - \exp\left(-\frac{e^z + e^{-z}}{2} + \frac{e^z - e^{-z}}{2}\right)\Bigg\} \\
&= \frac{1}{2}\Big\{\exp\left(\cosh z + \sinh z\right) - \exp\left(-\cosh z - \sinh z\right) \\
&\qquad + \exp\left(\cosh z - \sinh z\right) - \exp\left(-\cosh z + \sinh z\right)\Big\} \\
&= \frac{1}{2}\Big\{\exp\left(\cosh z\right)\exp\left(\sinh z\right) - \exp\left(-\cosh z\right)\exp(-\sinh z) \\
&\qquad + \exp\left(\cosh z\right)\exp\left(-\sinh z\right) - \exp\left(-\cosh z\right)\exp(\sinh z)\Big\} \\
&= \frac{1}{2}\Big\{\exp\left(\cosh z\right) - \exp\left(-\cosh z\right)\Big\}\Big\{\exp\left(\sinh z\right) + \exp(-\sinh z)\Big\} \\
&= 2\,\sinh(\cosh z)\,\cosh(\sinh z).
\end{aligned}$$

6. *Find all values z such that*

$$e^{3z+i} = -2.$$

   We write the equation as

$$e^{3x}e^{(3y+1)i} = e^{3z+i} = -2 = 2e^{\pi i}$$

   so that

$$3x = \ln 2 \qquad \text{and} \qquad 3y + 1 = \pi + 2n\pi, \qquad (n \in \mathbf{Z});$$

   and so

$$z = \frac{1}{3}\ln 2 + \frac{1}{3}\left(\pi - 1 + 2n\pi\right)i, \qquad (n \in \mathbf{Z}).$$

7. *Show that*

$$\log(i^{1/3}) = \frac{1}{3}\log i.$$

The logarithm is a multiple valued function. First, the three cube roots of $i = e^{\pi i/2}$ are

$$e^{(\pi/2+2\pi n)i/3} \quad (n \in \mathbf{Z})$$

so that the numbers corresponding to $n = 0, 1, 2$ are the distinct roots

$$e^{\pi i/6}, \quad e^{5\pi i/6}, \quad e^{3\pi i/2},$$

Then

$$\begin{aligned}
\log\left(e^{\pi i/6}\right) &= \ln 1 + \left(\frac{1}{6} + 2n\right)\pi i, \quad (n \in \mathbf{Z}),\\
\log\left(e^{5\pi i/6}\right) &= \ln 1 + \left(\frac{5}{6} + 2n\right)\pi i, \quad (n \in \mathbf{Z}),\\
\log\left(e^{3\pi i/2}\right) &= \ln 1 + \left(\frac{3}{2} + 2n\right)\pi i, \quad (n \in \mathbf{Z}).
\end{aligned}$$

Since $\frac{5}{6} = \frac{1}{6} + \frac{2}{3}$ and $\frac{3}{2} = \frac{1}{6} + 2 \cdot \frac{2}{3}$, their union is

$$\log\left(i^{1/3}\right) = \left(\frac{1}{6} + \frac{2n}{3}\right)\pi i, \quad (n \in \mathbf{Z}).$$

On the other hand,

$$\log i = \ln 1 + \left(\frac{1}{2} + 2n\right)\pi i, \quad (n \in \mathbf{Z})$$

so that

$$\frac{1}{3}\log i = \left(\frac{1}{6} + \frac{2n}{3}\right)\pi i, \quad (n \in \mathbf{Z})$$

which is the same as $\log\left(i^{1/3}\right)$.

8. *Verify the following relation for the inverse function.*

$$\operatorname{arccoth} z = \frac{1}{2}\log\frac{z+1}{z-1}.$$

$\operatorname{arccoth} z$ are the solutions w of the equation

$$z = \coth w = \frac{\cosh w}{\sinh w} = \frac{e^w + e^{-w}}{e^w - e^{-w}} = \frac{e^{2w}+1}{e^{2w}-1}.$$

Solving for e^{2w} yields

$$e^{2w} + 1 = z(e^{2w} - 1)$$

or

$$e^{2w} = \frac{z+1}{z-1}$$

Finally, taking logarithm gives the solution.

9. *Show that for different choices of* z_1 *and* z_2 *the following expression may or may not be valid:*

$$\operatorname{Log}\frac{z_1}{z_2} = \operatorname{Log} z_1 - \operatorname{Log} z_2.$$

The issue is whether or not the differnce of arguments remains in the domain of the principal value. Thus if we have $z_k = r_k e^{i\Theta_k}$ then $\operatorname{Log} z_k = \log z_k$ in the branch $-\pi < \Theta_k \le \pi$. Subtracting arguments in this range implies

$$-2\pi < \Theta_1 - \Theta_2 < 2\pi$$

which may put them out of range for the principal value if $\Theta_1 - \Theta_2 \le -\pi$ or $\Theta_1 - \Theta_2 > \pi$. For example if $z_1 = 1 + i = \sqrt{2}e^{\pi i/4}$ and $z_2 = -1 + i = \sqrt{2}e^{3\pi i/4}$ we have

$$\operatorname{Log}\frac{z_1}{z_2} = \operatorname{Log}(-i) = -\frac{\pi}{2}i = \left[\ln 2 + \frac{\pi}{4}i\right] - \left[\ln 2 + \frac{3\pi}{4}i\right] = \operatorname{Log} z_1 - \operatorname{Log} z_2.$$

On the other hand, if $z_1 = 1 - i = \sqrt{2}e^{-\pi i/4}$ and $z_2 = -1 + i = \sqrt{2}e^{3\pi i/4}$ we have

$$\operatorname{Log}\frac{z_1}{z_2} = \operatorname{Log}(-1) = \pi i \ne -\pi i = \left[\ln 2 - \frac{\pi}{4}i\right] - \left[\ln 2 + \frac{3\pi}{4}i\right] = \operatorname{Log} z_1 - \operatorname{Log} z_2.$$

10. *Find* $(1+i)^{3i}$ *and its principal value.*

$1 + i = \sqrt{2}e^{(1/4+2n)\pi i}$ for $n \in \mathbf{N}$. Its powers are thus

$$\begin{aligned}(1+i)^{3i} = e^{3i\log(1+i)} &= \exp\left(3i\left\{\ln\sqrt{2} + \left(\frac{1}{4} + 2n\right)\pi i\right\}\right), \quad (n \in \mathbf{Z})\\ &= \exp\left(-\frac{3\pi}{4} + 6n\pi\right)\exp\left(\frac{3\ln 2}{2}i\right), \quad (n \in \mathbf{Z})\end{aligned}$$

Its principal value corresponds to $N = 0$ or

$$\text{P. V. } (1+i)^{3i} = \exp\left(-\frac{3\pi}{4}\right)\exp\left(\frac{3\ln 2}{2}i\right)$$

11. *Find the roots of the equation* $\cos z = 3$.

Expressing the real and imaginary parts we find

$$\cos z = \cos x\cosh y - i\sin x\sinh y = 3$$

which reduces to two real equations

$$\cos x\cosh y = 3 \qquad \text{and} \qquad \sin x\sinh y = 0.$$

The second says either $\sinh y = 0$ which implies $y = 0$ or $\sin x = 0$ which implies

$$x = n\pi, \qquad (n \in \mathbf{N}).$$

If $y = 0$ then the first equation becomes

$$\cos x\cosh y = \cos x\cosh 0 = \cos x = 3$$

which has no real solution because $|\cos x| \le 1$. Thus we have to solve the first equation in the second case $x = n\pi$,

$$\cos x\cosh y = \cos n\pi\cosh y = (-1)^n\cosh y = 3.$$

But $\cosh y \geq 1$ so that this equation only has solutions if n is even, in which case $y = \cosh^{-1} 3$, namely, real s that solves

$$3 = \cosh s = \frac{1}{2}(e^s + e^{-s}).$$

Equivalently,

$$0 = e^{2s} - 6e^s + 1$$

or

$$e^s = \frac{6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 1}}{2} = 3 \pm 2\sqrt{2}.$$

Noting that $3 + 2\sqrt{2}$ and $3 - 2\sqrt{2}$ are reciprocals, we may write all solutions as

$$z = 2n\pi \pm i \ln(3 + 2\sqrt{2}), \qquad (n \in \mathbf{Z}).$$

12. *Show that* $\tanh z = -i \tan(iz)$ *and* $\dfrac{d}{dz} \tanh z = \operatorname{sech}^2 z$.

We use the identities $\sinh z = -i \sin iz$ and $\cosh z = \cos iz$. Then

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{-i \sin iz}{\cos iz} = -i \tan iz.$$

Also, by the definitions of $\sinh z$ and $\cosh z$ and the quotient rule

$$\begin{aligned}\frac{d}{dz} \tanh z = \frac{d}{dz} \frac{\sinh z}{\cosh z} = \frac{d}{dz} \frac{e^z - e^{-z}}{e^z + e^{-z}} &= \frac{(e^z + e^{-z})(e^z + e^{-z}) - (e^z + e^{-z})(e^z - e^{-z})}{(e^z + e^{-z})^2} \\ &= \frac{\left(e^{2z} + 2 + e^{-2z}\right) - \left(e^{2z} - 2 + e^{-2z}\right)}{(e^z + e^{-z})^2} = \frac{4}{(e^z + e^{-z})^2} = \frac{1}{\cosh^2 z} = \operatorname{sech}^2 z.\end{aligned}$$

13. *Evaluate the following integral, assuming* $\Re\mathrm{e}\, z < 0$.

$$\int_0^\infty e^{zt}\, dt$$

This is an improper integral. Observing that $\dfrac{d}{dt} \dfrac{e^{zt}}{z} = e^{zt}$ we obtain

$$\int_0^\infty e^{zt}\, dt = \lim_{T \to \infty} \int_0^T e^{zt}\, dt = \lim_{T \to \infty} \left[\frac{e^{zt}}{z} \right]_0^T = \lim_{T \to \infty} \left[\frac{e^{zT}}{z} - \frac{1}{z} \right] = -\frac{1}{z}.$$

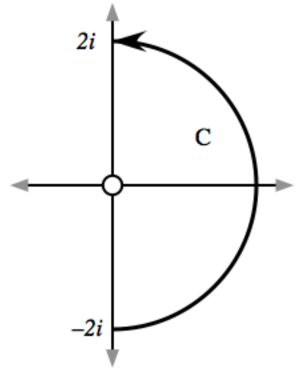
For $z = x + iy$, the latter limit depends on the estimate

$$|e^{zT}| = |e^{xT} e^{iyT}| = e^{xT}$$

which tends to zero as $T \to \infty$ because $x = \Re\mathrm{e}\, z < 0$.

14. *let C denote the right half of the circle $|z| = 2$ in the counterclockwise direction. Consider the two given parametric representations $z_1(\theta)$ and $z_2(y)$ of C. Find the reparameterization $\theta = \phi(y)$ such that $z_2(y) = z_1[\phi(y)]$ and check that ϕ has positive derivative.*

$$z_1(\theta) = 2e^{i\theta}, \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$
$$z_2(y) = \sqrt{4 - y^2} + iy, \qquad (-2 \le y \le 2).$$



Putting the first representation in rectangular coordinates gives

$$x = 2\cos\theta \qquad \text{and} \qquad y = 2\sin\theta.$$

Equating the ratio we find

$$\frac{y}{\sqrt{4-y^2}} = \frac{y}{x} = \frac{2\sin\theta}{2\cos\theta} = \tan\theta$$

or

$$\theta = \phi(y) = \operatorname{Atn}\left(\frac{y}{\sqrt{4-y^2}}\right), \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right).$$

To see that $z_2(y) = z_1[\phi(y)]$, we note that $\cos(\operatorname{Atn} t) = 1/\sqrt{1+t^2}$ and $\sin(\operatorname{Atn} t) = t/\sqrt{1+t^2}$, so

$$\Re\mathrm{e}\, z_1(\phi(y)) = 2\cos\left(\operatorname{Atn}\left(\frac{y}{\sqrt{4-y^2}}\right)\right) = \frac{2}{\sqrt{1+\dfrac{y^2}{4-y^2}}} = \frac{2\sqrt{4-y^2}}{\sqrt{4}} = \Re\mathrm{e}\, z_2(y)$$

and

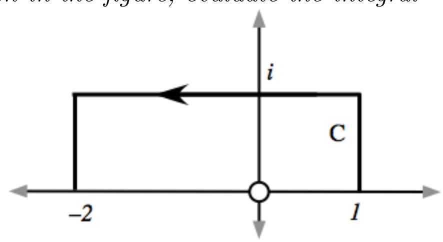
$$\Im\mathrm{m}\, z_1(\phi(y)) = 2\sin\left(\operatorname{Atn}\left(\frac{y}{\sqrt{4-y^2}}\right)\right) = \frac{\dfrac{2y}{\sqrt{4-y^2}}}{\sqrt{1+\dfrac{y^2}{4-y^2}}} = \frac{2y}{\sqrt{4}} = \Im\mathrm{m}\, z_2(y).$$

For $-2 < y < 2$ we find

$$\phi'(y) = \frac{1}{1+\left(\dfrac{y}{\sqrt{4-y^2}}\right)^2} \cdot \frac{\sqrt{4-y^2} + \dfrac{y^2}{\sqrt{4-y^2}}}{\left(\sqrt{4-y^2}\right)^2} = \frac{1}{4} \cdot \frac{4}{\sqrt{4-y^2}} > 0$$

15. *Using the contour C shown in the figure, evaluate the integral*

$$\int_C \Re\mathrm{e}(e^{iz})\, dz$$



The function is the derivative in $\mathbf{C}$

$$F'(z) = \frac{d}{dz}\frac{e^{iz}}{i} = e^{iz}$$

Thus the integral is independent of path, and its value may be computed using the end points $z_1 = 1$ and $z_2 = -2$ to give

$$I = \int_C \Re\mathrm{e}(e^{iz})\, dz = \Re\mathrm{e}\left(\int_C f(z)\, dz\right) = \Re\mathrm{e}\left(F(z_2) - F(z_1)\right).$$

But

$$F(z_2) - F(z_1) = \left[\frac{e^{iz}}{i}\right]_1^{-2} = \frac{e^{-2i}}{i} - \frac{e^{i}}{i} = -i\big[\cos(-2) + i\sin(-2) - \cos 1 - i\sin 1\big]$$

so

$$I = -\sin 2 - \sin 1.$$

16. *Let C denote the positively oriented circle $|z| = 5$ about the origin. For the branch at angle $\alpha \in \mathbf{R}$, with its corresponding* $\log z = \ln r + i\theta$ *where $\alpha < \theta < \alpha + 2\pi$, find*

$$\int_C z^{-2/3}\, dz.$$

We have $z(\theta) = 5e^{i\theta}$ for $\alpha < \theta < \alpha + 2\pi$ so $dz = 5ie^{i\theta}\, d\theta$. Also

$$z^{-2/3}\, z'(\theta) = \exp\left(-\frac{2}{3}\log(z(\theta))\right) \cdot 5ie^{i\theta} = 5i\exp\left(-\frac{2}{3}\ln 5 + \frac{1}{3}i\theta\right)$$

As this extends continuously to the interval $\alpha \le \theta \le \alpha + 2\pi$, we may integrate using real antiderivatives as usual

$$\begin{aligned}\int_C z^{-2/3}\, dz &= \int_\alpha^{\alpha+2\pi} z^{-2/3}\, z'(\theta)\, d\theta = i\sqrt[3]{5}\left[\frac{e^{i\theta/3}}{\frac{i}{3}}\right]_\alpha^{\alpha+2\pi} \\ &= 3\sqrt[3]{5}\left[e^{i(\alpha+2\pi)/3} - e^{i\alpha/3}\right] = 3\sqrt[3]{5}\left[e^{2\pi i/3} - 1\right] e^{i\alpha/3}.\end{aligned}$$

17. *Let C_N denote the boundary of the square formed by the lines*

$$x = \pm\left(N + \frac{1}{2}\right)\pi \qquad \textit{and} \qquad y = \pm\left(N + \frac{1}{2}\right)\pi$$

where N is a positive integer and C_N is given the counterclockwise orientation. Show that the integral tends to zero as $N \to \infty$.

$$\int_{C_N} \frac{dz}{z^2 \sin z}$$

We bound the absolute value. The length of the square is $L_N = 4(2N+1)\pi$. To bound the integrand, we observe that

$$\sin z = \sin(x+iy) = \sin x\, \cos iy + \cos x\, \sin iy = \sin x\, \cosh y + i \cos x \sinh y.$$

It follows that

$$\begin{aligned} |\sin z|^2 &= \sin^2 x\, \cosh^2 y + \cos^2 x\, \sinh^2 y \\ &= \sin^2 x(1 + \sinh^2 y) + \cos^2 x\, \sinh^2 y \\ &= \sin^2 x + \sinh^2 y \geq \sin^2 x \end{aligned}$$

so that on the vertical lines,

$$|\sin z| \geq |\sin x| = \left|\sin\left(\pm\left[N + \frac{1}{2}\right]\pi\right)\right| = 1.$$

The estimate also showed that

$$|\sin z| \geq |\sinh y|$$

so that on the horizontal lines, since $\sinh y$ is increasing in y and $N \geq 0$,

$$|\sin z| \geq |\sinh y| = \left|\sinh\left(\pm\left[N + \frac{1}{2}\right]\pi\right)\right| = \sinh\left(\left[N + \frac{1}{2}\right]\pi\right) \geq \sinh\left(\frac{\pi}{2}\right) \geq 1.$$

So on all four sides of the square $|\sin z| \geq 1$. Thus we may estimate the integrand

$$\left|\frac{1}{z^2 \sin z}\right| = \frac{1}{|z|^2\, |\sin z|} \leq \frac{1}{\left(N + \frac{1}{2}\right)^2 \pi^2} = M_N$$

because $|z| \geq |x|$ on the verticals and $|z| \geq |y|$ on the horizontals, which are both equal $\left(N + \frac{1}{2}\right)\pi$.

Then the contour integral may be estimated

$$\left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \leq L_N M_N = \frac{4(2N+1)\pi}{\left(N + \frac{1}{2}\right)^2 \pi^2} = \frac{16}{(2N+1)\,\pi}.$$

Thus the value of the integral tends to zero as $N \to \infty$.

18. *Find the integral, where the integrand is computed using the principal branch and C is any contour from $z = 3$ to $z = -2$ that, except for its end points, lies above the real axis.*

$$\int_C z^{-2i}\, dz$$

The restriction on C means that in polar coordinates, for $z \in C$ we have $r > 0$ and $0 \le \theta \le \pi$. In this range

$$z^{-2i} = \exp\left(-2i \operatorname{Log} z\right), \qquad (0 \le \theta \le \pi).$$

For this range of θ's, $\operatorname{Log} z = \log z$ where the second logarithm has the branch cut along the negative imaginary axis $R_{-\pi/2}$ so $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. The function has an antiderivative on the domain $D = \mathbf{C} - R_{-\pi/2}$, namely

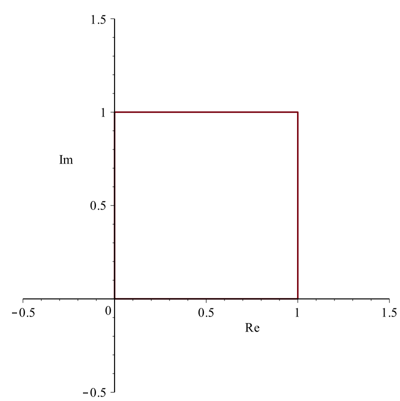
$$F(z) = \frac{z^{-2i+1}}{-2i+1} = \frac{1+2i}{5}\exp((-2i+1)\log z).$$

Hence the integral is found by evaluating the antiderivative function at the end points $z_1 = 3 = 3e^{0\cdot i}$ and $z_2 = -2 = 2e^{\pi i}$ of the contour $C \subset D$

$$\begin{aligned}\int_C z^{-2i}\, dz &= F(-2) - F(3) = \frac{1+2i}{5}\Big(\exp[(-2i+1)(\ln 2 + \pi i)] - \exp[(-2i+1)\ln 3]\Big)\\ &= -\frac{1+2i}{5}\left(2e^{2\pi}e^{-2i\ln 2} + 3e^{-2i\ln 3}\right).\end{aligned}$$

19. *Using the contour C which is a unit square with corners 0, 1, $1+i$ and i shown in the figure, evaluate the integral*

$$\int_C \cos\left(3 + \frac{1}{z-3}\right) dz$$



Observe that the integrand function

$$f(z) = \cos\left(3 + \frac{1}{z-3}\right)$$

is the composition of an entire function with a rational function which has only one singularity at $z = 3$. So $f(z)$ is analytic in the punctured plane, $\mathcal{D} = \mathbf{C} - \{3\}$. Since C is a simple, closed contour in $\mathcal{D}$, by the Cauchy-Goursat Theorem, the integral is zero.

20. *Show that the integral on the real axis*

$$I = \int_{-\infty}^{\infty} \frac{x^2 + 4 - 3i}{(x + 2i)^4}\, dx = 0.$$

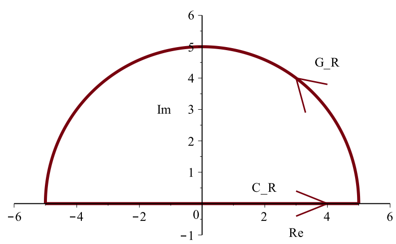
Let

$$f(z) = \frac{z^2 + 4 - 3i}{(z + 2i)^4}.$$

Since f decays quadratically in $|z|$, the function is integrable on $\mathbf{R}$. For $R > 0$, letting C_R denote the line segment from $-R$ to R, we have

$$I = \lim_{R\to\infty} \int_{C_R} \frac{z^2 + 4 - 3i}{(z + 2i)^4}\, dz.$$

Observe that the only singularity of the rational function $f(z)$ is at $z = -2i$ so it is analytic in $\mathcal{D} = \mathbf{C} - \{-2i\}$. Let G_R denote the arc of the circle $|z| = R$ in the upper half plane.



Thus $C_R + G_R$ is a simple closed contour in $\mathcal{D}$ so that, by the Cauchy Goursat Theorem,

$$0 = \int_{C_R + G_R} f(z)\, dz = \int_{C_R} f(z)\, dz + \int_{G_R} f(z)\, dz.$$

We show that the limit of the second integral vanishes as $R \to \infty$, thus so does the first, so $I = 0$.

We estimate the second integral. The length of the contour is $L_R = \pi R$. Also for $z \in \Gamma_R$ and $R > 3$, by the triangle inequalty

$$|f(z)| = \frac{|z^2 + 4 - 3i|}{|z + 2i|^4} \le \frac{|z|^2 + |4 - 3i|}{\big||z| - |3i|\big|^4} = \frac{R^2 + 5}{(R - 3)^4} = M_R$$

Hence

$$\left| \int_{G_R} f(z)\, dz \right| \le L_R \cdot M_R = \frac{\pi R (R^2 + 5)}{(R - 3)^4}$$

which tends to zero as $R \to \infty$, completing the argument.