Math 3160 § 1. **First Midterm Exam** Name: Practice Problems
Treibergs February 5, 2015

1. *Calculate the following expressions*

$$a)\quad \frac{(1+2i)^3-(1-i)^3}{(3+2i)^3-(2+i)^3} \qquad b)\quad (a+b\omega+c\omega^2)(a+b\omega^2+c\omega) \quad \text{where } \omega=\frac{-1+\sqrt{3}\,i}{2}.$$

Since

$$\begin{aligned}(1+2i)^3 &= 1+3(2i)+3(2i)^2+(2i)^3 = 1+6i-12-8i = -11-2i\\ (1-i)^3 &= 1+3(-i)+3(-i)^2+(-i)^3 = 1-3i-3+i = -2-2i\\ (3+2i)^3 &= 3^3+3\cdot 3^2(2i)+3\cdot 3(2i)^2+(2i)^3 = 27+54i-36-8i = -9+46i\\ (2+i)^3 &= 2^3+3\cdot 2^2 i+3\cdot 2i^2+i^3 = 8+12i-6-i = 2+11i\end{aligned}$$

we get

$$\begin{aligned}\frac{(1+2i)^3-(1-i)^3}{(3+2i)^3-(2+i)^3} &= \frac{-11-2i-(-2-2i)}{-9+46i-(2+11i)} = \frac{-9}{-11+35i}\\ &= \frac{(-9)(-11-35i)}{(-11+35i)(-11-35i)} = \frac{-117+27+26i+315i}{11^2+35^2} = \frac{99+315i}{1346}.\end{aligned}$$

Also $\omega = e^{2\pi i/3}$ so $\omega+\omega^2=-1$ and $\omega^3=1$. Thus

$$\begin{aligned}(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) &= a^2+ab\omega^2+ac\omega+ba\omega+b^2\omega^3+bc\omega^2+ca\omega^2+cb\omega^4+c^2\omega^3\\ &= (a^2+b^2+c^2)\omega^3+(ab+ac+bc)(\omega+\omega^2) = a^2+b^2+c^2-ab-ac-bc\end{aligned}$$

2. *For complex numbers α and β, prove using only the definition of the operations, modulus and conjugation that*

$$\overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta}, \qquad |\,\Re\mathrm{e}\,\alpha| \le |\alpha| \le |\,\Re\mathrm{e}\,\alpha| + |\,\Im\mathrm{m}\,\alpha|$$

Write $\alpha=(a,b)$ and $\beta=(c,d)$, we have

$$\begin{aligned}\overline{\alpha\beta} &= \overline{(a,b)(c,d)}\\ &= \overline{(ac-bd,ad+bc)}\\ &= (ac-bd,-ad-bc)\\ &= (ac-(-b)(-c),a(-d)+(-b)c)\\ &= (a,-b)(c,-d)\\ &= \overline{(a,b)}\,\overline{(c,d)}\\ &= \bar{\alpha}\,\bar{\beta}.\end{aligned}$$

Also

$$|\,\Re\mathrm{e}\,\alpha| = |\,\Re\mathrm{e}(a,b)| = |a| = \sqrt{a^2+0^2} \le \sqrt{a^2+b^2} = |(a,b)| = |\alpha|$$

and

$$\begin{aligned}|\alpha| = |(a,b)| = \sqrt{a^2+b^2} &= \sqrt{|a|^2+|b|^2} \le \sqrt{|a|^2+2|a||b|+|b|^2}\\ &= \sqrt{(|a|+|b|)^2} = |a|+|b| = |\,\Re\mathrm{e}\,\alpha| + |\,\Im\mathrm{m}\,\alpha|.\end{aligned}$$

3. *Prove the identity and interpret it geometrically.*

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2).$$

We have

$$\begin{aligned}|z_1+z_2|^2+|z_1-z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)}+(z_1-z_2)\overline{(z_1-z_2)}\\ &= z_1\bar z_1+z_1\bar z_2+z_2\bar z_1+z_2\bar z_2+z_1\bar z_1-z_1\bar z_2-z_2\bar z_1+z_2\bar z_2\\ &= 2z_1\bar z_1+2z_2\bar z_2=2(|z_1|^2+|z_2|^2).\end{aligned}$$

It is the parallelogram law. It says that in the parallelogram with vertices 0, z_1, z_2 and z_1+z_2 the sums of the squares of the lengths of the diagonals equals twice the sum of the squares of the lengths of the sides.

4. *Prove that* $|(1+i)z^3+iz|<\frac{3}{4}$ *if* $|z|<\frac{1}{2}$.

$$\begin{aligned}|(1+i)z^3+iz| &= |((1+i)z^2+i)z|\\ &= \big|(1+i)z^2+i\big|\,|z|\\ &\le \big(|(1+i)z^2|+|i|\big)|z|\\ &= \big(|1+i|\,|z|^2+1\big)|z|\\ &= (\sqrt{2}|z|^2+1)|z|\\ &\le \left(\frac{\sqrt{2}}{4}+1\right)\frac{1}{2}\\ &= \frac{\sqrt{2}+4}{8}<\frac{2+4}{8}=\frac{3}{4}.\end{aligned}$$

5. *By a purely geometric argument, prove that* $|z-1|\le\big||z|-1\big|+|z|\,|\arg z|$.

Let γ be the arc of the circle centered at the origin and radius $|z|$ starting at $|z|$ on the real axis and ending at z. Its length, $|z||\arg z|$ is greater than the distance between endpoints. Thus the triangle inequality for the three points 1, $|z|$ and z in the plane gives

$$|z-1|=\operatorname{dist}(1,z)\le\operatorname{dist}(1,|z|)+\operatorname{dist}(|z|,z)\le\operatorname{dist}(1,|z|)+\operatorname{length}(\gamma)=\big||z|-1\big|+|z|\,|\arg z|.$$

6. *Solve the following equations:*

$$a)\quad |z|-z=1+2i;\qquad b)\quad |z|+z=2+i.$$

a) Write $z=x+iy$ so that the equation becomes

$$\sqrt{x^2+y^2}-x-iy=1+2i$$

so that $y=-2$ and so $\sqrt{x^2+(-2)^2}-x=1$. Adding and squaring,

$$x^2+4=(1+x)^2=1+2x+x^2$$

so $3=2x$ or $x=\frac{3}{2}$. One checks that $z=\frac{3}{2}-2i$ solves the equation.

b) This time the equation becomes

$$\sqrt{x^2+y^2}+x+iy=2+i$$

so that $y=1$ and so $\sqrt{x^2+1^2}+x=2$. Subtracting and squaring,

$$x^2+1=(2-x)^2=4-4x+x^2$$

so $-3=-4x$ or $x=\frac{3}{4}$. One checks that $z=\frac{3}{4}+i$ solves the equation.

7. *What are the loci of points z that satisfy the following relations*

$$a)\quad |z-2|+|z+2|=5, \qquad b)\quad 0<\Re\mathrm{e}(iz)<1, \qquad c)|z|=\Re\mathrm{e}\, z+1.$$

a) This equation says $\mathrm{dist}(z,2)+\mathrm{dist}(z,-2)=5$. This is an ellipse with center at the origin, with major axis along the x axis and minor along the y-axis with major and minor radii $a=\frac{5}{2}$ and $b=\sqrt{(\frac{5}{2})^2-2^2}=\frac{3}{2}$.

b) $\Re\mathrm{e}(iz)=\Re\mathrm{e}(ix-y)=-y$ so the locus is the horizontal slab $\{x+iy : -1<y<0\}$.

c) The equation is $\sqrt{x^2+y^2}=x+1$ so $x^2+y^2=x^2+2x+1$ so $y^2=2x+1$ which is the parabola opening in the positive x-direction given by $x=\frac{1}{2}y^2-\frac{1}{2}$.

8. *Prove that every complex number of unit modulus (except $z=-1$) can be represented in the form*

$$z=\frac{1+it}{1-it}$$

where t is a real number.

One observes that $|1+iy|=|1-it|$ so that all such z satisfy

$$|z|=\frac{|1+iy|}{|1-it|}=1,$$

thus have unit modulus. It remains to see if the $\mathrm{Arg}\, z$ takes all values in $(-\pi,\pi)$. But $\mathrm{Arg}(1+it)=\mathrm{Atn}\, t$ (triangle has base 1 and height t) which takes all values in $(-\frac{\pi}{2},\frac{\pi}{2})$ as t runs through all real values. Also $\mathrm{Arg}(1-it)=-\mathrm{Atn}\, t$ which also takes all values in $(-\frac{\pi}{2},\frac{\pi}{2})$ as t runs through all real values. Thus $\arg z=\mathrm{Arg}(1+it)-\mathrm{Arg}(1-it)=\mathrm{Atn}\, t-(-\mathrm{Atn}\, t)=2\,\mathrm{Atn}\, t$ which takes all values in $(-\pi,\pi)$ as t runs through all reals. Note, too, that since $2\,\mathrm{Atn}\, t$ is a strictly increasing function, there is a one-to-one correspondence between the real numbers t and the complex numbers in the circle about the origin excluding -1.

9. *Suppose that $|z_1|=|z_2|=|z_3|>0$. Show that*

$$\arg\frac{z_3-z_2}{z_3-z_1}=\frac{1}{2}\arg\frac{z_2}{z_1}.$$

This is just the geometric statement that the angle between two points on a circle viewed from a third point on the circle is half the angle viewed from the center. We may suppose that $z_1\neq z_3$ and $z_2\neq z_3$ for the left side to be defined. Thus write $z_1=re^{ia}$, $z_2=e^{ib}$ and $z_3=e^{ic}$ where a,b,c are real numbers such that $c<a,b<c+2\pi$. It follows that

$$\frac{1}{2}\arg\frac{z_2}{z_1}=\frac{1}{2}\arg\frac{re^{ib}}{re^{ia}}=\frac{b-a}{2}.$$

Observe that

$$\arg\frac{z_3-z_2}{z_3-z_1}=\arg\frac{re^{ic}-re^{ib}}{re^{ic}-re^{ia}}=\arg\frac{e^{ic}(1-e^{i(b-c)})}{e^{ic}(1-e^{i(b-c)})}=\arg\frac{e^{i(b-c)}-1}{e^{i(a-c)}-1}$$

Using the trigonometric identities,

$$\begin{aligned}
e^{i\theta} - 1 &= \cos\theta - 1 + i\sin\theta \\
&= \cos\left(2\cdot\tfrac{\theta}{2}\right) - 1 + i\sin\left(2\cdot\tfrac{\theta}{2}\right) \\
&= \cos^2\left(\tfrac{\theta}{2}\right) - \sin^2\left(\tfrac{\theta}{2}\right) - 1 + 2i\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right) \\
&= -2\sin^2\left(\tfrac{\theta}{2}\right) + 2i\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right) \\
&= 2\sin\left(\tfrac{\theta}{2}\right)\left[-\sin\left(\tfrac{\theta}{2}\right) + i\cos\left(\tfrac{\theta}{2}\right)\right] \\
&= 2\sin\left(\tfrac{\theta}{2}\right)\left[\cos\left(\tfrac{\theta}{2}+\tfrac{\pi}{2}\right) + i\sin\left(\tfrac{\theta}{2}+\tfrac{\pi}{2}\right)\right] \\
&= 2\sin\left(\tfrac{\theta}{2}\right)e^{i\left(\frac{\theta}{2}+\frac{\pi}{2}\right)}
\end{aligned}$$

Since $0 < a - c < 2\pi$ and $0 < b - c < 2\pi$, it follows that

$$\arg\frac{z_3 - z_2}{z_3 - z_1} = \arg\frac{e^{i(b-c)} - 1}{e^{i(a-c)} - 1} = \arg\frac{2\sin\left(\frac{b-c}{2}\right)e^{i\left(\frac{b-c}{2}+\frac{\pi}{2}\right)}}{2\sin\left(\frac{a-c}{2}\right)e^{i\left(\frac{a-c}{2}+\frac{\pi}{2}\right)}} = \left(\frac{b-c}{2}+\frac{\pi}{2}\right) - \left(\frac{a-c}{2}+\frac{\pi}{2}\right) = \frac{b-a}{2}.$$