Math 3160 § 1. **Final Exam** Name: Practice Problems

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Problems taken from the Midterm exam given Feb. 27, 1998 and the Final exam given Mar. 17, 1998. Half of our final exam will focus on material since the last midterm. The other half will be comprehensive. Questions on this review cover mainly problems since the last mid.

(F.1.) *Find a Laurent series for* $f(z) = \dfrac{1}{z-i} + \dfrac{1}{z+2i}$ *which converges in the annulus* $1 < |z| < 2$ *centered at zero.*

(F.2.) *Suppose* c *is the contour consisting of straight line segments from* $2+2i$ *to* $-2+2i$ *to* $-2-2i$ *to* $2-2i$ *and back to* $2+2i$. *Find both* $\displaystyle\int_c \frac{\sin z\,dz}{(z-i)^2}$ *and* $\displaystyle\int_c \frac{\sin z\,dz}{(z-3i)^2}$. *Write the answers in the form* $x+iy$. *(Explanation required!)*

(F.3.) Suppose we wanted to have a single valued (analytic) cube root function f(z) = $z^{1/3}$ *defined on a domain* $\mathbf{C}-\{0\}$. *(That is a function* $f : \mathbf{C}-\{0\} \to \mathbf{C}$ *such that* $f(z)^3 = z$ *for all* $z \neq 0$.*) Show that there is no single valued analytic cube root defined on all of* $\mathbf{C}-\{0\}$. *Show that if for some fixed* ϑ_0, *in the domain* D *that misses the points* $re^{i\vartheta_0}$ *for* $r \ge 0$, *then there are three distinct single valued analytic cube root functions on* D.

(F.4a.) *Find* $\displaystyle\operatorname*{Res}_{z=0}\left(\frac{1}{1-\exp(z^2)}\right)$.

(F.4b.) *Show that if* z_0 *is a pole of order* $m \ge 1$ *for* $f(z)$, *then* $\lim_{z\to z_0} f(z) = \infty$

(F.5.) *Find* $I = \displaystyle\int_0^\infty \frac{dx}{x^4+4}$ *using contour integration. Find a contour and formulate an expression for the improper integral* I *involving a limit of an integral over the contour. Account for all pieces of your contour and explain why the "garbage terms" go to zero. [Hint:* $z^4+4 = (z+1+i)(z+1-i)(z-1+i)(z-1-i)$ *]*

(F.6a.) *Suppose* $f(z)$ *is analytic in a punctured disk around* z_0 *and has a simple pole at* z_0. *Explain why*

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z)$$

(F.6b.) *Assuming that the series converges near zero, find* $\displaystyle\operatorname*{Res}_{z=0}\frac{1}{z^2+a_3z^3+a_4z^4+\cdots}$

(F.7.) *Find two Laurent series for the function* $f(z) = \dfrac{1}{(z-1)^2}$ *centered at zero, one convergent near* $z_0 = 0$, *the other convergent far away from* $z_0 = 0$. *What are the precise regions of convergence for your two series?*

(F.8.) *Find* $I_1 = \displaystyle\int_0^\infty \frac{x\,dx}{1+x^3}$. *Find a contour and formulate an expression for the improper integral* I_1 *involving a limit of an integral over the contour. Account for all pieces of your contour and explain why the "garbage terms" go to zero. Hints:* $\displaystyle\operatorname*{Res}_{z=z_0}\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = c_{-1}$,

$$\operatorname*{Res}_{z=-1}\frac{z}{1+z^3} = -\frac{1}{3}, \qquad \operatorname*{Res}_{z=e^{\pi i/3}}\frac{z}{1+z^3} = \frac{1-\sqrt{3}i}{6}, \qquad \operatorname*{Res}_{z=-e^{\pi i/3}}\frac{z}{1+z^3} = \frac{1+\sqrt{3}i}{6}.$$

(F.9.) *Find* $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)\,dx}{1+x^2}$ *using complex analysis. You have to justify why the "garbage terms" go to zero.*

More Problems

(E.1.) *Find two Laurent series for* $h(z) = \dfrac{1}{(1-z)^2}$ *centered at* $z_0 = i$*. For what values of* z *do each of the series converge?*

(E.2.) *Find the Taylor's series for* $f(z) = \dfrac{1}{1+z^2}$ *about* $z_0 = 2i$*. For which values of* z *does your series converge?*

(E.3.) *Find the following integrals using Residues: (a.)* $\displaystyle\int_0^{2\pi} \frac{d\,\vartheta}{4+\sin^2\vartheta}$, *(b.)* $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)\,dx}{x^2-x+2}$

(E.4.) *Compute the inverse Laplace transform of* $F(s) = \dfrac{1}{s^3+8}$, *ie., for* $t>0$*, find the Principal Value integral taken along the infinite contour* $\mathbf{c}$*, the straight line* $\Re\mathrm{e}\, z = \gamma$ *oriented in the upward direction, where* $\gamma > 0$ *is a constant large enough so that all the poles of* $F(s)$ *occur to the left of c.* $f(t) = \dfrac{1}{2\pi i}\,\mathrm{P.V.}\displaystyle\int_{\mathbf{c}} \frac{e^{st}\,ds}{s^3+8}$

(E.5.) *Suppose that* $f(z)$ *is an entire function whose value is bounded over the entire plane by* $|f(z)| \le 99$ *for all* $z \in \mathbf{C}$*. Show that* $f(z)$ *is everywhere constant.*

Solutions of the final.

(F.1.) *Find a Laurent series for* $f(z) = \dfrac{1}{z-i} + \dfrac{1}{z+2i}$ *which converges in the annulus* $1 < |z| < 2$ *centered at zero.*

The poles are i and $-2i$. We take the Maclaurin series of the second term which converges in $|z| < 2$ and the Laurent series for the first term that converges for $|z| > 1$. Then the Laurent series of $f(z)$ is the sum of the two series.

$$\frac{1}{z+2i} = \frac{1}{2i\left(1+\frac{z}{2i}\right)} = \frac{1}{2i}\left(1 - \left(\frac{z}{2i}\right) + \left(\frac{z}{2i}\right)^2 - \left(\frac{z}{2i}\right)^3 + \cdots\right) = \frac{1}{2i} + \frac{z}{4} - \frac{z^2}{8i} - \frac{z^3}{16} + \cdots$$

$$\frac{1}{z-i} = \frac{1}{z\left(1-\frac{i}{z}\right)} = \frac{1}{z}\left(1 + \frac{i}{z} + \left(\frac{i}{z}\right)^2 + \cdots\right) = \frac{1}{z} + \frac{i}{z^2} - \frac{1}{z^3} - \frac{i}{z^4} + \cdots$$

$$f(z) = \cdots - \frac{i}{z^4} - \frac{1}{z^3} + \frac{i}{z^2} + \frac{1}{z} + \frac{1}{2i} + \frac{z}{4} - \frac{z^2}{8i} - \frac{z^3}{16} + \cdots$$

(F.2.) *Suppose* $\mathbf{c}$ *is the contour consisting of straight line segments from* $2+2i$ *to* $-2+2i$ *to* $-2-2i$ *to* $2-2i$ *and back to* $2+2i$*. Find both* $I_1 = \displaystyle\int_c \frac{\sin z\,dz}{(z-i)^2}$ *and* $I_2 = \displaystyle\int_c \frac{\sin z\,dz}{(z-3i)^2}$ *. Write the answers in the form* $x+iy$*.*

The singularity of $f(z) = \dfrac{\sin z}{(z-i)^2}$ is $z_0 = i$, inside the contour. Thus

$$I_1 = \int_{\mathbf{c}} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z) = 2\pi i \left.\frac{d}{dz}\right|_{z=z_0} \sin z = 2\pi i\cos(z_0) = 2\pi i\cos(i) = 2\pi i\cosh(1).$$

On the other hand, the only pole of $g(z) = \dfrac{\sin z}{(z-3i)^2}$ is $z_0 = 3i$ which is outside the contour $\mathbf{c}$. Hence g is analytic on and inside the contour, making $I_2 = 0$ by the Cauchy-Goursat Theorem.

(F.3.) *Suppose we wanted to have a single valued (analytic) cube root function $f(z) = z^{1/3}$ defined on a domain $\mathbf{C} - \{0\}$. (That is a function $f : \mathbf{C} - \{0\} \to \mathbf{C}$ such that $f(z)^3 = z$ for all $z \neq 0$.) Show that there is no single valued analytic cube root defined on all of $\mathbf{C} - \{0\}$. Show that if for some fixed ϑ_0 the domain D misses the points $re^{i\vartheta_0}$ for $r \geq 0$, then there are three distinct single valued analytic cube root functions on D.*

Suppose there is an analytic function on $\mathbf{C} - \{0\}$ such that $(f(z))^3 = z$ for all $z \neq 0$. Then, for all points $0 < |z| \leq 1$ we must have $|f(z)| \leq 1$. It follows that $z_0 = 0$ is a removeable singularity. Thus, the function can be given a Maclaurin series for $0 < |z| < 1$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

which extends to 0 as an analytic function, which we still call $f(z)$. Since $f(0) = 0$, let the order of the zero $z_0 = 0$ be $m \in \mathbf{N}$ so that there is an analytic function $g(z)$ defined near zero so that $g(0) \neq 0$ and so that $f(z) = z^m g(z)$. Now, using the fact that the cube is known,

$$z = (f(z))^3 = (z^m g(z))^3 = z^{3m}(g(z))^3$$

where g^3 is an analytic function defined near zero which doesn't vanish near zero. Since the order of zeros must be the same, we get $1 = 3m$. But there is no integer solution for this equation hence there is no such f.

Now, suppose that we make a branch cut. For $D = \{re^{i\vartheta} : r > 0,\ \vartheta_0 < \vartheta < \vartheta_0 + 2\pi\}$ we define three functions for $k = 0, 1, 2$,

$$f_k(re^{i\vartheta}) = r^{1/3} e^{(\vartheta + 2\pi k)i/3}$$

These are analytic on D, and are cube roots

$$\left(f_k(re^{i\vartheta})\right)^3 = \left(r^{1/3} e^{(\vartheta+2\pi k)i/3}\right)^3 = re^{(\vartheta+2\pi k)i} = re^{i\vartheta}.$$

Moreover, they are distinct functions since for $k = 0, 1, 2$,

$$f_k(re^{\vartheta_0+\pi}) = r^{1/3}e^{(\vartheta_0+\pi+2\pi k)i/3} = r^{1/3}e^{(\vartheta_0+\pi)i/3}, r^{1/3}e^{(\vartheta_0+3\pi)i/3}, r^{1/3}e^{(\vartheta_0+5\pi)i/3}$$

which are all distinct.

(F.4a.) *Find* $R = \operatorname*{Res}_{z=z_0} \left(\dfrac{1}{1-\exp(z^2)}\right)$.

The residue is c_{-1} in the Laurent expansion $f(z) = \sum_{-\infty}^{\infty} c_n z^n$ in some region $0 < |z| < R$. Expanding the power series,

$$\begin{aligned}
\frac{1}{1-\exp(z^2)} &= \frac{1}{1-\left(1+z^2+\frac12 z^4+\frac16 z^6+\cdots\right)} = -\frac{1}{z^2+\frac12 z^4+\frac16 z^6+\cdots} \\
&= -\frac{1}{z^2}\left(\frac{1}{1+\frac12 z^2+\frac16 z^4+\cdots}\right) = -\frac{1}{z^2}\left(1-\frac12 z^2+\frac{1}{12}z^4+\cdots\right) \\
&= -\frac{1}{z^2}+\frac12-\frac{1}{12}z^2+\cdots
\end{aligned}$$

using long division. Thus $R = c_{-1} = 0$.

(F.4b.) *Show that if* z_0 *is a pole of order* $m \geq 1$ *for* $f(z)$, *then* $\lim_{z\to z_0} f(z) = \infty$

If $f(z)$ has a pole of order m at z_0 then $f(z) = (z-z_0)^{-m}\phi(z)$ where ϕ is an analytic function near z_0 such that $\phi(z_0) \neq 0$. Now, since ϕ is continuous near z_0 as it is analytic, we may find an $\varepsilon > 0$ so that $|\phi(z_0) - \phi(z)| < \frac{1}{2}|\phi(z_0)|$ whenever $R = |z - z_0| < \varepsilon$. It follows from the reverse triangle inequality that

$$|\phi(z)| = |\phi(z_0) - (\phi(z_0) - \phi(z))| \geq |\phi(z_0)| - |\phi(z) - \phi(z_0)| \geq |\phi(z_0)| - \frac{1}{2}|\phi(z_0)| = \frac{1}{2}|\phi(z_0)| > 0$$

whenever $|z - z_0| < \varepsilon$. It follows that the limit is infinite because when $|z - z_0| < \varepsilon$,

$$|f(z)| = \left|\frac{\phi(z)}{(z-z_0)^m}\right| \geq \frac{|\phi(z)|}{|z-z_0|^m} \geq \frac{\frac{1}{2}|\phi(z_0)|}{R^m} \to \infty$$

whenever $R \to 0$ which means whenever $z \to z_0$.

(F.5.) *Find* $I = \displaystyle\int_0^\infty \frac{dx}{x^4+4}$ *using contour integration. Find a contour and formulate an expression for the improper integral* I *involving a limit of an integral over the contour. Account for all pieces of your contour and explain why the "garbage terms" go to zero. [Hint:* $z^4 + 4 = (z+1+i)(z+1-i)(z-1+i)(z-1-i)$ *]*

As the integrand $f(z) = (z^4+4)^{-1}$ is even, we have

$$2I = \text{P. V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty} \left\{ \oint_{\mathbf{c}_R} f(z)\,dz - \int_{\mathbf{c}''} f(z)\,dz \right\} = \frac{\pi}{4} + 0$$

where $\mathbf{c}_R$ is a closed contour consisting of a straight line segment from $-R$ to R followed by the semicircular arc $\mathbf{c}''$ given by $z = Re^{it}$ for $0 < t < \pi$. For $R > \sqrt{2}$, two of the four poles lie within $\mathbf{c}_R$, namely $z_0 = \pm 1 + i$. It follows that the contour integral equals

$$\begin{aligned}
\oint_{\mathbf{c}_R} f(z)\,dz &= 2\pi i \left(\operatorname*{Res}_{z=-1+i} f(z) + \operatorname*{Res}_{z=1+i} f(z) \right) \\
&= 2\pi i \left(\left.\frac{1}{(z+1+i)(z-1+i)(z-1-i)}\right|_{-1+i} + \left.\frac{1}{(z+1+i)(z+1-i)(z-1+i)}\right|_{1+i} \right) \\
&= 2\pi i \left(\frac{1}{(2i)(-2+2i)(-2)} + \frac{1}{(2+2i)(2)(2i)} \right) = \frac{\pi}{4}.
\end{aligned}$$

Now let's check that the "garbage term" goes to zero. Intuitively, $f(z)$ decays like R^{-4} as $|z| = R \to \infty$, whereas the length pf $\mathbf{c}''$ is πR. Hence the garbage integral decays like πR^{-3} as $R \to \infty$. To make a rigorous argument, we observe from the reverse triangle inequality that $|R^4 e^{4ti} + 4| \geq |R^4 e^{4ti}| - |4| = R^4 - 4$. Now estimating the garbage integral, putting $z = Re^{it}$ gives

$$\left| \int_{\mathbf{c}''} \frac{dz}{z^4+4} \right| = \left| \int_0^\pi \frac{Rie^{it}\,dt}{R^4 e^{4it} + 4} \right| \leq \int_0^\pi \frac{|Rie^{it}|\,dt}{|R^4 e^{4it} + 4|} \leq \int_0^\pi \frac{R\,dt}{R^4 - 4} = \frac{\pi R}{R^4 - 4} \to 0$$

as $R \to \infty$.

(F.6a.) *Suppose $f(z)$ is analytic in a punctured disk around z_0 and has a simple pole at z_0. Explain why*

$$\operatorname*{Res}_{z=z_0 0} f(z) = \lim_{z\to z_0}(z-z_0)f(z)$$

A simple pole means that there is only one term in the principal part. Expressing the function as a Laurent series in some $0 < |z-z_0| < R$,

$$f(z) = \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

The residue is defined to be $\operatorname*{Res}_{z=z_0} f(z) = b_1$. Now, computing the limit,

$$\begin{aligned}\lim_{z\to z_0}(z-z_0)f(z) &= \lim_{z\to z_0}(z-z_0)\left(\frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right)\\ &= \lim_{z\to z_0}\bigl(b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + \cdots\bigr) = b_1\end{aligned}$$

(F.6b.) *Assuming that the series converges near zero, find* $R = \operatorname*{Res}_{z=0} \dfrac{1}{z^2 + a_3z^3 + a_4z^4 + \cdots}$

The residue is the b_1 term of the Laurent expansion in some $0 < |z| < R$. By long division

$$\begin{aligned}\frac{1}{z^2 + a_3z^3 + a_4z^4 + \cdots} &= \frac{1}{z^2}\left(\frac{1}{1 + a_3z + a_4z^2 + \cdots}\right)\\ &= \frac{1}{z^2}\Bigl(1 - a_3z + (a_3^2 - a_4)z^2 + \cdots\Bigr) = \frac{1}{z^2} - \frac{a_3}{z} + (a_3^2 - a_4) + \cdots\end{aligned}$$

Thus $R = b_1 = -a_3$.

(F.7.) *Find two Laurent series for the function $f(z) = \dfrac{1}{(z-1)^2}$ centered at zero, one convergent near $z_0 = 0$, the other convergent far away from $z_0 = 0$. What are the precise regions of convergence for your two series?*

$f(z)$ has a pole at $z_0 = 1$. The Maclaurin series for $(1-z)^{-1}$ converges exactly if $|z| < 1$. Hence the same is true for its square

$$f(z) = \frac{1}{1-z}\cdot\frac{1}{1-z} = \bigl(1 + z + z^2 + z^3 + \cdots\bigr)\cdot\bigl(1 + z + z^2 + z^3 + \cdots\bigr) = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

which converges exactly when $|z| < 1$. (The series blows up at $z = 1$ and converges for $|z| < 1$ by the ratio test.)

Similarly, the Laurent Series converging for exactly $|z| > 1$ is the product

$$\begin{aligned}f(z) &= \left(\frac{1}{z-1}\right)^2 = \frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z}}\right)^2 = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)^2\\ &= \frac{1}{z^2}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right) = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \cdots\end{aligned}$$

(*e.g.*, the series diverges at $z = 1$ and converges for $1/|z| < 1$ by the ratio test.)

(F.8.) *Find* $I_1 = \displaystyle\int_0^\infty \frac{x\,dx}{1+x^3}$. *Find a contour and formulate an expression for the improper integral* I_1 *involving a limit of an integral over the contour. Account for all pieces of your contour and explain why the "garbage terms" go to zero. Hints:* $\displaystyle\operatorname*{Res}_{z=z_0} \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = c_{-1}$,

$$\operatorname*{Res}_{z=-1} \frac{z}{1+z^3} = -\frac{1}{3}, \qquad \operatorname*{Res}_{z=e^{\pi i/3}} \frac{z}{1+z^3} = \frac{1-\sqrt{3}i}{6}, \qquad \operatorname*{Res}_{z=e^{-\pi i/3}} \frac{z}{1+z^3} = \frac{1+\sqrt{3}i}{6}.$$

Observe that $z^3+1 = (z+1)(z-\omega)(z-\bar\omega)$ where $\omega = e^{\pi i/6}$ so $\omega^3 = \bar\omega^3 = -1$. Now let $\mathbf{c}_R$ denote the contour consisting of the line segment from 0 to R, then the circular arc $\mathbf{c}''$ given by $z = Re^{it}$ for $0 < t < 2\pi/3$, and then the straight line segment $\mathbf{c}'$ from $R\omega^2$ to 0. By parameterizing $z = \omega^2 t$ we find

$$\int_{\mathbf{c}'} \frac{z\,dz}{z^3+1} = -\int_0^R \frac{(\omega^2 t)\,\omega^2\,dt}{\omega^6 t^3+1} = -\omega^4 \int_0^R \frac{t\,dt}{t^3+1} = \omega \int_0^R \frac{t\,dt}{t^3+1}.$$

Thus, we may find the integral by adding the real axis part and the $\mathbf{c}'$ part to get

$$(1+\omega)I_1 = \lim_{R\to\infty} \left(\oint_{\mathbf{c}_R} \frac{z\,dz}{z^3+1} - \int_{\mathbf{c}''} \frac{z\,dz}{z^3+1} \right)$$

Now, the only pole inside the contour is $z_0 = \omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ so that the first integral is given by

$$\oint_{\mathbf{c}_R} \frac{z\,dz}{z^3+1} = 2\pi i \operatorname*{Res}_{z=e^{\pi i/3}} \frac{z}{1+z^3} = 2\pi i \frac{1-\sqrt{3}i}{6} = \frac{2\pi(\sqrt{3}+i)}{6} = \frac{2\pi(1+\omega)}{3\sqrt{3}}$$

Thus if the "garbage term" goes to zero, $I_1 = \frac{2\pi}{3\sqrt{3}}$. The intuition for the vanishing of the garbage terms is that the function decays like $R/R^3 = 1/R^2$ on $\mathbf{c}''$ and the length of $\mathbf{c}''$ is $2\pi R/3$ so that the integral is bounded by about $2\pi/(3R) \to 0$ as $R \to \infty$. To make a rigorous estimate, observe that by the reverse triangle inequality, $|R^3 e^{3iy}+1| \ge |R^3 e^{3it}| - |1| = R^3 - 1$. Then, parameterizing by $z = Re^{it}$, we have

$$\left| \int_{\mathbf{c}''} \frac{z\,dz}{z^3+1} \right| = \left| \int_0^{2\pi/3} \frac{(Re^{it})\,Rie^{it}\,dt}{R^3 e^{3it}+1} \right| \le \int_0^{2\pi/3} \frac{|Re^{it}|\,|Rie^{it}|\,dt}{|R^3 e^{3it}+1|} \le \int_0^{2\pi/3} \frac{R^2\,dt}{R^3-1} = \frac{2\pi R^2}{3(R^3-1)} \to 0$$

as $R \to \infty$.

(F.9.) *Find* $\displaystyle\int_{-\infty}^\infty \frac{\sin(x)\,dx}{1+x^2}$ *using complex analysis. You have to justify why the "garbage terms" go to zero.*

Observe that $\Im\mathrm{m}(e^{ix}) = \sin x$. Taking $\mathbf{c}_R$ to be the contour consisting of a straight line segment from $-R$ to R and the arc $\mathbf{c}''$ given by $z = Re^{it}$ for $0 < t < \pi$, the integral is

$$\begin{aligned} I = \mathrm{P.V.} \int_{-\infty}^\infty \frac{\sin(x)\,dx}{1+x^2} &= \lim_{R\to\infty} \int_{-R}^R \frac{\sin(x)\,dx}{1+x^2} \\ &= \lim_{R\to\infty} \Im\mathrm{m} \left(\oint_{\mathbf{c}_R} \frac{\exp(iz)\,dz}{1+z^2} - \int_{\mathbf{c}''} \frac{\exp(iz)\,dz}{1+z^2} \right) = 0 \end{aligned}$$

The first integral is computed from the residue formula. The second is a "garbage" term that tends to zero. The only pole within $\mathbf{c}_R$ is $z_0 = i$. Thus for $R > 1$,

$$\oint_{\mathbf{c}_R} \frac{\exp(iz)\,dz}{1+z^2} = 2\pi i \left(\operatorname*{Res}_{z=i} \frac{\exp(iz)}{(z-i)(z+i)} \right) = 2\pi i \left(\left. \frac{\exp(iz)}{z+i} \right|_{z=i} \right) = \pi e^{-1}$$

which is real.

We have

$$\exp(iz) = \exp(iRe^{it}) = \exp(iR(\cos t + i\sin t)) = \exp(-R\sin t + iR\cos t) = e^{-R\sin t}e^{iR\cos t},$$

so that because in this range, $\sin t \geq 0$, we have $|\exp(iz)| \leq 1$. Also, by the reverse triangle inequality, $|R^2e^{2it}+1| \geq |R^2e^{2it}| - |1| = R^2 - 1$. Intuitively, the garbage term vanishes because the integrand decays like $1/R^2$, the length of $\mathbf{c}''$ grows like R, so the integral decays like $1/R$ as $R \to \infty$. The rigorous check is as follows:

$$\left|\int_{\mathbf{c}''} \frac{\exp(iz)\,dz}{1+z^2}\right| = \left|\int_0^\pi \frac{\exp(iRe^{it})\,Rie^{it}\,dt}{R^2e^{2it}+1}\right| \leq \int_0^\pi \frac{1\cdot|Rie^{it}|\,dt}{|R^2e^{2it}+1|}$$
$$\leq \int_0^\pi \frac{R\,dt}{R^2-1} = \frac{\pi R}{R^2-1} \to 0$$

as $R \to \infty$.

Solutions of the additional problems.

(E.1.) *Find two Laurent series for $h(z) = \dfrac{1}{(1-z)^2}$ centered at $z_0 = i$. For what values of z do each of the series converge?*

$h(z)$ is analytic at i, and the closest pole to i is at 1, thus the series will converge in $|z-i| < |1-i| = \sqrt{2}$ exactly. The easiest way to find the series is to square the geometric sum. On $|z-i| < \sqrt{2}$ we have

$$\begin{aligned} h(z) &= \left(\frac{1}{1-z}\right)^2 = \left(\frac{1}{1-i-(z-i)}\right)^2 = \frac{1}{(1-i)^2}\left(\frac{1}{1-\frac{z-i}{1-i}}\right)^2 \\ &= \frac{i}{2}\left(1 + \frac{z-i}{1-i} + \left[\frac{z-i}{1-i}\right]^2 + \cdots\right)^2 = \frac{i}{2}\left(1 + 2\frac{z-i}{1-i} + 3\left[\frac{z-i}{1-i}\right]^2 + \cdots\right) \\ &= \frac{i}{2} + i\frac{z-i}{1-i} + \frac{3i}{2}\left[\frac{z-i}{1-i}\right]^2 + \cdots \end{aligned}$$

For the other region $|z-i| > \sqrt{2}$ we factor the other way.

$$\begin{aligned} h(z) &= \left(\frac{1}{z-1}\right)^2 = \left(\frac{1}{z-i-(1-i)}\right)^2 = \frac{1}{(z-i)^2}\left(\frac{1}{1-\frac{1-i}{z-i}}\right)^2 \\ &= \frac{1}{(z-i)^2}\left(1 + \frac{1-i}{z-i} + \left[\frac{1-i}{z-i}\right]^2 + \cdots\right)^2 = \frac{1}{(z-i)^2}\left(1 + 2\frac{1-i}{z-i} + 3\left[\frac{1-i}{z-i}\right]^2 + \cdots\right) \\ &= \frac{1}{(z-i)^2} + \frac{2(1-i)}{(z-i)^3} + \frac{3(1-i)^2}{(z-i)^4} + \cdots \end{aligned}$$

We can check that the series diverges when $z = 1$ and converges if $|1-i|/|z-i| < 1$ by the ratio test.

(E.2.) *Find the Taylor's series for $f(z) = \dfrac{1}{1+z^2}$ about $z_0 = 2i$. For which values of z does your series converge?*

The poles of $f(z)$ are $\pm i$, so that the series will converge for $|z-2i| < |i-2i| = 1$ and diverge otherwise. We can find the coefficients using the Taylor's formula, or simply multiplying series.

The first converges in $|z-2i|<1$ and the second in $|z-2i|<3$. The product converges in the smaller disk.

$$\begin{aligned}
f(z) &= \left(\frac{1}{i-z}\right)\left(\frac{1}{i+z}\right) = \left(\frac{1}{-i-(z-2i)}\right)\left(\frac{1}{3i+(z-2i)}\right)\\
&= \frac{1}{(-i)(3i)}\left(\frac{1}{1-i(z-2i)}\right)\left(\frac{1}{1-\frac{i}{3}(z-2i)}\right)\\
&= \frac{1}{3}\left(1+i(z-2i)+i^2(z-2i)^2+\cdots\right)\left(1+\frac{i}{3}(z-2i)+\frac{i^2}{3^2}(z-2i)^2+\cdots\right)\\
&= \frac{1}{3}\left(1+\frac{4}{3}i(z-2i)+\frac{13}{3^2}i^2(z-2i)^2+\cdots\right)\\
&= \frac{1}{3}+\frac{4i}{9}(z-2i)-\frac{13}{27}(z-2i)^2+\cdots
\end{aligned}$$

(E.3.) *Find the following integrals using Residues:*

(a.) $I_1 = \displaystyle\int_0^{2\pi}\frac{d\vartheta}{4+\sin^2\vartheta}$, *(b.)* $I_2 = \displaystyle\int_{-\infty}^{\infty}\frac{\cos(x)\,dx}{x^2-x+2}$

The trick for the first integral is to notice that this one is exactly a contour integral. By setting $z=e^{i\vartheta}$ for real ϑ we have $z=\cos\vartheta+i\sin\vartheta$ and $\frac{1}{z}=e^{-i\vartheta}=\cos\vartheta-i\sin\vartheta$. Hence $\frac{1}{2i}\left(z+\frac{1}{z}\right)=\sin\vartheta$ and $dz=ie^{i\vartheta}d\vartheta=iz\,d\vartheta$. Thus the integral becomes a contour integral

$$I_1=\oint_{|z|=1}\frac{\frac{dz}{iz}}{4-\frac{1}{4}\left(z+\frac{1}{z}\right)^2}=\frac{4}{i}\oint_{|z|=1}\frac{z\,dz}{16z^2-(z^2+1)^2}=4i\oint_{|z|=1}\frac{z\,dz}{(z^2-4z+1)(z^2+4z+1)}$$

The poles are therefore $z=\pm2\pm\sqrt{3}$. Only $z_1=2-\sqrt{3}$ and $z_2=-2+\sqrt{3}$ fall within $|z|=1$. Therefore we may use the residues

$$\begin{aligned}
I_1 &= -8\pi\left(\operatorname*{Res}_{z=2-\sqrt{3}}\frac{z}{(z^2-4z+1)(z^2+4z+1)}+\operatorname*{Res}_{z=-2+\sqrt{3}}\frac{z}{(z^2-4z+1)(z^2+4z+1)}\right)\\
&= -8\pi\left(\left.\frac{z}{(z-2-\sqrt{3})(z^2+4z+1)}\right|_{z=2-\sqrt{3}}+\left.\frac{z}{(z^2-4z+1)(z+2+\sqrt{3})}\right|_{z=-2+\sqrt{3}}\right)\\
&= -8\pi\left(-\frac{1}{16\sqrt{3}}-\frac{1}{16\sqrt{3}}\right)=\frac{\pi}{\sqrt{3}}
\end{aligned}$$

For the second integral, observe that $\Re\mathrm{e}(e^{ix})=\cos x$. Taking $\mathbf{c}_R$ to be the contour consisting of a straight line segment from $-R$ to R and the arc $\mathbf{c}''$ given by $z=Re^{it}$ for $0<t<\pi$, the integral becomes

$$\begin{aligned}
I_2 &= \mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\cos(x)\,dx}{x^2-x+2}=\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos(x)\,dx}{x^2-x+2}\\
&= \lim_{R\to\infty}\Re\mathrm{e}\left(\oint_{\mathbf{c}_R}\frac{\exp(iz)\,dz}{z^2-z+2}-\int_{\mathbf{c}''}\frac{\exp(iz)\,dz}{z^2-z+2}\right)
\end{aligned}$$

The first integral is computed from the residue formula. The second is a "garbage" term that tends to zero. The poles are $z_0=\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$. The only pole within $\mathbf{c}_R$ is $z_0=\frac{1}{2}+\frac{\sqrt{7}}{2}i$. Thus for

$R > \sqrt{2}$,

$$\oint_{\mathbf{c}_R} \frac{\exp(iz)\,dz}{z^2 - z + 2} = 2\pi i \operatorname*{Res}_{z=\frac{1}{2}+\frac{\sqrt{7}}{2}i} \left(\frac{\exp(iz)}{z^2 - z + 2} \right)$$
$$= 2\pi i \left(\left. \frac{\exp(iz)}{z - \frac{1}{2} + \frac{\sqrt{7}}{2} i} \right|_{z=\frac{1}{2}+\frac{\sqrt{7}}{2}i} \right) = \frac{2\pi e^{-\sqrt{7}/2} \left(\cos \frac{1}{2} + i \sin \frac{1}{2} \right)}{\sqrt{7}}.$$

Thus, $I_2 = \dfrac{2\pi e^{-\sqrt{7}/2} \cos(1/2)}{\sqrt{7}}$, provided that the garbage term goes to zero.

We have

$$\exp(iz) = \exp(iRe^{it}) = \exp(iR(\cos t + i \sin t)) = \exp(-R \sin t + iR \cos t) = e^{-R \sin t} e^{iR \cos t},$$

so that because in this range, $\sin t \geq 0$, we have $|\exp(iz)| \leq 1$. Also, by the reverse triangle inequality, $|R^2 e^{2it} - Re^{it} + 2| \geq |R^2 e^{2it}| - |Re^{it} - 2| \geq R^2 - R - 2$. Intuitively the garbage term vanishes because the integrand decays like $1/R^2$, the length of $\mathbf{c}''$ grows like R, so the integral decays like $1/R$ as $R \to \infty$. The rigorous check is as follows:

$$\left| \int_{\mathbf{c}''} \frac{\exp(iz)\,dz}{z^2 - z + 2} \right| = \left| \int_0^\pi \frac{\exp(iRe^{it})\, Rie^{it}\,dt}{R^2 e^{2it} - Re^{it} + 2} \right| \leq \int_0^\pi \frac{|\exp(iRe^{it})|\,|Rie^{it}|\,dt}{|R^2 e^{2it} - Re^{it} + 2|}$$
$$\leq \int_0^\pi \frac{1 \cdot R\,dt}{R^2 - R - 2} = \frac{\pi R}{R^2 - R - 2} \to 0$$

as $R \to \infty$.

(E.4.) *Compute the inverse Laplace transform of* $F(s) = \dfrac{1}{s^3 + 8}$ *ie. for* $t > 0$, *find the Principal value integral taken along the infinite contour* $\mathbf{c}$, *the straight line* $\Re z = \gamma$ *oriented in the upward direction, where* $\gamma > 0$ *is a constant large enough so that all the poles of* $F(s)$ *occur to the left of c.*

$$f(t) = \frac{1}{2\pi i} \,\mathrm{P.V.} \int_{\mathbf{c}} \frac{e^{st}\,ds}{s^3 + 8}$$

Let $\omega = 2e^{\pi i/3} = 1 + \sqrt{3}i$ so that $\omega^3 = \bar{\omega}^3 = -8$. Taking $\mathbf{c}_R$ to be the contour consisting of a straight line segment from $\gamma - Ri$ to $\gamma + Ri$ and the arc $\mathbf{c}''$ given by $z = Re^{i\tau}$ for $\alpha = \arccos(\gamma/R) < \tau < 2\pi - \arccos(\gamma/R) = \beta$. Hence, the integral

$$2\pi i f(t) = \mathrm{P.V.} \int_{\mathbf{c}} \frac{e^{tz}\,dz}{z^3 + 8} = \lim_{R \to \infty} \int_{\gamma - Ri}^{\gamma + Ri} \frac{e^{tz}\,dz}{z^3 + 8} = \lim_{R \to \infty} \left(\oint_{\mathbf{c}_R} \frac{e^{tz}\,dz}{z^3 + 8} - \int_{\mathbf{c}''} \frac{e^{tz}\,dz}{z^3 + 8} \right)$$

The first integral is computed from the residue formula. The second is a "garbage" term that tends to zero. The poles are $z_0 = \omega, \bar{\omega}, -2$. Thus for $R > 2$,

$$\oint_{\mathbf{c}_R} \frac{e^{tz}\,dz}{z^3 + 8} = 2\pi i \left(\operatorname*{Res}_{z=\omega} \frac{e^{tz}}{z^3 + 8} + \operatorname*{Res}_{z=\bar{\omega}} \frac{e^{tz}}{z^3 + 8} + \operatorname*{Res}_{z=-2} \frac{e^{tz}}{z^3 + 8} \right)$$
$$= 2\pi i \left(\left. \frac{e^{tz}}{(z - \bar{\omega})(z + 2)} \right|_{z=\omega} + \left. \frac{e^{tz}}{(z - \omega)(z + 2)} \right|_{z=\bar{\omega}} + \left. \frac{e^{tz}}{(z - \omega)(z - \bar{\omega})} \right|_{z=-2} \right)$$
$$= 2\pi i \left(\frac{e^{t\omega}}{(\omega - \bar{\omega})(\omega + 2)} + \frac{e^{t\bar{\omega}}}{(\bar{\omega} - \omega)(\bar{\omega} + 2)} + \frac{e^{-2t}}{(-2 - \omega)(-2 - \bar{\omega})} \right)$$

Thus, $f(t) = \dfrac{e^{t\omega}}{(\omega - \bar{\omega})(\omega + 2)} + \dfrac{e^{t\bar{\omega}}}{(\bar{\omega} - \omega)(\bar{\omega} + 2)} + \dfrac{e^{-2t}}{(-2 - \omega)(-2 - \bar{\omega})}$, provided that the garbage term goes to zero.

On this range of τ, we have $\Re\mathrm{e}(Re^{i\tau}) = R\cos\tau \le \gamma$.

$$\exp(tz) = \exp(tRe^{i\tau}) = \exp(tR(\cos\tau + i\sin\tau)) = e^{tR\cos\tau}e^{itR\sin\tau},$$

so that because in this range, $R\cos\tau \le \gamma$, we have $|\exp(tz)| \le e^{t\gamma}$. Also, by the reverse triangle inequality, $|R^3e^{3it}+8| \ge |R^3e^{3it}|-|8| = R^3-8$. Intuitively, the garbage term vanishes because the integrand decays like $e^{t\gamma}/R^3$, the length of $\mathbf{c}''$ grows like πR, so the integral decays like $\pi e^{t\gamma}/R^2$ as $R \to \infty$. The rigorous check is as follows. Note that $0 < \alpha < \Pi/2$ and $3\pi/2 < \beta < \pi$ so

$$\begin{aligned}\left|\int_{\mathbf{c}''} \frac{e^{tz}\,dz}{z^3+8}\right| &= \left|\int_\alpha^\beta \frac{\exp(tRe^{i\tau})\,Rie^{i\tau}\,d\tau}{R^3e^{3i\tau}+8}\right| \le \int_\alpha^\beta \frac{|\exp(tRe^{i\tau})|\,|Rie^{i\tau}|\,d\tau}{|R^3e^{3i\tau}+8|} \\ &\le \int_\alpha^\beta \frac{Re^{t\gamma}\,dt}{R^3-8} \le \frac{R\left(e^{\beta\gamma}-e^{\alpha\gamma}\right)}{\gamma(R^3-8)} \le \frac{R\left(e^{2\pi\gamma}-1\right)}{\gamma(R^3-8)} \to 0\end{aligned}$$

as $R \to \infty$.

(E.5.) *Suppose that $f(z)$ is an entire function whose value is bounded over the entire plane by $|f(z)| \le 99$ for all $z \in \mathbf{C}$. Show that $f(z)$ is everywhere constant.*

Since $f(z)$ is entire, it is analytic on the disk $|z| < R+1$ for all $R > 0$. Then the Maclaurin's expansion of f, which is defined on and inside the disk $|z| \le R$ is given by

$$f(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots$$

Using Cauchy's formula for the k-th coefficient, and the bound $|f(z)| \le 99$,

$$\begin{aligned}|a_k| &= \left|\frac{1}{2\pi i}\oint_{|z|=R} \frac{f(s)\,ds}{s^{k+1}}\right| = \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(Re^{it})\,Rie^{it}\,dt}{R^{k+1}e^{(k+1)it}}\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(Re^{it})|\,|Rie^{it}|\,dt}{R^{k+1}|e^{(k+1)it}|} \le \frac{1}{2\pi}\int_0^{2\pi} \frac{99\,dt}{R^k} = \frac{99}{R^k}\end{aligned}$$

Now, for $k \ge 1$, since R can be taken to be arbitrarily large, only $|a_k| = 0$ can be possible. It follows that $f(z) = a_0 + 0 + 0 + 0 + \cdots$, or f is a constant.