Math 3160 § 1. **Second Midterm Exam** Name: Sample Problems
Treibergs March 30, 2006

Midterm exams given Jan. 30 and Feb. 27, 1998.

(M.1.) *Using the fact that* $\cot z = \dfrac{\mathbf{i}\, e^{\mathbf{i}\, z} + \mathbf{i}\, e^{-\mathbf{i}\, z}}{e^{\mathbf{i}\, z} - e^{-i\, z}}$, *find* $\cot^{-1} z$.

(M.2.) *Suppose that* $f(z)$ *is analytic inside the unit circle* $|z| \le 1$ *and has the values* $f(e^{\mathbf{i}\theta}) = \cos(3\theta) + \mathbf{i}\sin^2\theta$ *for* $0 \le \theta < 2\pi$. *Find* $f(0)$.

(M.3.) *Find a parameterization for the circular contour* $\mathbf{c}$ *centered at* $\mathbf{i}$ *with radius* R. *Then compute the contour integral* $\displaystyle\oint_{\mathbf{c}} \frac{dz}{z - \mathbf{i}}$ *over* $\mathbf{c}$.

(M.4.) *Find all possible values of* $\mathbf{i}^{(1+\mathbf{i})}$. *What is the principal value?*

(M.5.) *Let* $\mathbf{c}$ *be the contour consisting of a straight line segment from* 8 *to* $\mathbf{i}$ *followed by the segment from* $\mathbf{i}$ *to* -8. *If* $z^{1/3}$ *is defined for* $z = re^{\mathbf{i}\theta}$ *by* $z^{1/3} = \sqrt[3]{r}\, e^{\mathbf{i}\,\theta/3}$, *find* $\displaystyle\int_{\mathbf{c}} z^{1/3} dz$. *Do the integral by first deforming the contour to a new contour which is easier to integrate. Explain why this deformation is possible.*

Solutions of the midterm.

(M.1.) Let $p = e^{iw}$. Then

$$z = \cot w = \frac{ie^{iw} + ie^{-iw}}{e^{iw} - e^{-iw}} = \frac{i\left(p + \frac{1}{p}\right)}{p - \frac{1}{p}} = \frac{i(p^2+1)}{p^2-1}.$$

Thus, solving for p^2, $z(p^2-1) = i(p^2+1)$ so $p^2(z-i) = z+i$ so

$$p^2 = \frac{z+i}{z-i}$$

so

$$p = \left(\frac{z+i}{z-i}\right)^{\frac{1}{2}}$$

is double valued. Finally, since $\frac{1}{2}\log z = \log(z^{1/2})$ for all $z \ne 0$, (see problem 99[5] of Brown & Churchill, 9th. ed.)

$$w = \cot^{-1} z = -i\log p = \frac{i}{2}\log\left(\frac{z-i}{z+i}\right),$$

which has infinitely many values at all points. A single valued analytic function results when specific branches are chosen for square root and logarithm.

(M.2.) Let $\mathbf{c}$ be the unit circle parameterized by $z(t) = e^{\mathbf{i}\, t}$, $0 \le t < 2\pi$ so that $dz = \mathbf{i}\, e^{\mathbf{i}\, t}\, dt$. On $\mathbf{c}$, $f(z(t)) = \cos 3t + \mathbf{i}\sin^2 t$. Choosing $z_0 = 0$ we find using the Cauchy Integral formula,

$$f(z_0) = \frac{1}{2\pi\mathbf{i}} \oint_{\mathbf{c}} \frac{f(z)\, dz}{z - z_0}$$

so

$$f(0) = \frac{1}{2\pi\mathbf{i}} \oint_{\mathbf{c}} \frac{f(z)\, dz}{z} = \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \frac{\left(\cos 3t + \mathbf{i}\sin^2 t\right)\, \mathbf{i}\, e^{\mathbf{i}\, t}\, dt}{e^{\mathbf{i}\, t}}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cos 3t + \mathbf{i}\sin^2 t\, dt = \frac{1}{2\pi}(0 + \pi\mathbf{i}) = \frac{\mathbf{i}}{2}.$$

(M.3.) Let $\mathbf{c}$ be the circle of radius $R > 0$ parameterized by $z(t) = \mathbf{i} + Re^{\mathbf{i}\,t}$ for $0 \le t < 2\pi$ so that $dz = \mathbf{i}\,R\,e^{\mathbf{i}\,t}\,dt$. Then

$$\oint_{\mathbf{c}} \frac{dz}{z-\mathbf{i}} = \int_0^{2\pi} \frac{\mathbf{i}\,R\,e^{\mathbf{i}\,t}\,dt}{Re^{\mathbf{i}\,t}} = \int_0^{2\pi} \mathbf{i}\;dt = 2\pi\mathbf{i}.$$

(M.4.) The logarithm $\log\mathbf{i} = \left\{\ln|\mathbf{i}| + \left(\frac{\pi}{2} + 2\pi k\right)\mathbf{i} : k \in \mathbf{Z}\right\} = \left\{\left(\frac{\pi}{2} + 2\pi k\right)\mathbf{i} : k \in \mathbf{Z}\right\}$. By using logs and exponentials,

$$\begin{aligned}
\mathbf{i}^{(1+\mathbf{i})} &= \exp\left((1+\mathbf{i})\log\mathbf{i}\right)\\
&= \exp\left((1+\mathbf{i})\left\{\left(\tfrac{\pi}{2} + 2\pi k\right)\mathbf{i} : k \in \mathbf{Z}\right\}\right)\\
&= \left\{\exp\left((\tfrac{\pi}{2} - 2\pi k)(\mathbf{i} - 1)\right) : k \in \mathbf{Z}\right\}\\
&= \left\{e^{-\frac{\pi}{2} + 2\pi k}\left(\cos(\tfrac{\pi}{2} - 2\pi k) + \mathbf{i}\sin(\tfrac{\pi}{2} - 2\pi k)\right) : k \in \mathbf{Z}\right\}\\
&= \left\{\mathbf{i}\,e^{-\frac{\pi}{2} + 2\pi k} : k \in \mathbf{Z}\right\}
\end{aligned}$$

Thus the principal value is gotten using $\operatorname{Log}\mathbf{i} = \frac{\pi}{2}\mathbf{i}$ so P.V. $\mathbf{i}^{(1+\mathbf{i})} = \mathbf{i}\,e^{-\pi/2}$.

(M.5.) By taking the branch cut along the negative t-axis, we have analyticity in $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Then the branch of cube root $z^{1/3} = \sqrt[3]{r}\,e^{\mathbf{i}\,\theta/3}$ is analytic off the negative imaginary axis. Thus if we let the new contour $\mathbf{c}'$ be the semicircle in the upper half plane centered at the origin with radius $R = 8$ parameterized by $z(t) = 8e^{\mathbf{i}\,t}$ for $0 \le t \le \pi$, then $z^{1/3}$ is analytic on the closed curve $\mathbf{c}' - \mathbf{c}$ and in the region bounded by it. Thus the integrals are the same. Using $dz = 8\mathbf{i}\,e^{\mathbf{i}\,t}\,dt$,

$$\begin{aligned}
\int_{\mathbf{c}} z^{1/3} dz = \int_{\mathbf{c}'} z^{1/3} dz = \int_0^{\pi} \sqrt[3]{8}\,e^{\mathbf{i}\,t/3}\,8\mathbf{i}\,e^{\mathbf{i}\,t}\,dt &= 16\mathbf{i}\int_0^{\pi} \cos(\tfrac{4t}{3}) + \mathbf{i}\,\sin(\tfrac{4t}{3})\,dt\\
= 16\mathbf{i}\left(\tfrac{3\sqrt{3}}{8} + \tfrac{9}{8}\mathbf{i}\right) = -18 + 6\sqrt{3}\,\mathbf{i}.
\end{aligned}$$

Some extra practice problems.

(E.1.) *Without computing derivatives, estimate the norm of* $\dfrac{d^{10} f}{d\,z^{10}}(2\mathbf{i})$ *where* $f(z) = \dfrac{z}{1+z^2}$

Get the estimate from the Cauchy Integral Formula for the tenth derivative on the circle $\mathbf{c}$ about $2\mathbf{i}$ of radius $R = \frac{1}{2}$ where $f(z)$ is analytic. That is because the function has singularities at $\pm\mathbf{i}$. Thus $|z| = |z - 2\mathbf{i} + 2\mathbf{i}| \le |z - 2\mathbf{i}| + |2\mathbf{i}| = \frac{5}{2}$. Also $|z| = |2\mathbf{i} + z - 2\mathbf{i}| \ge |2\mathbf{i}| - |z - 2\mathbf{i}| = \frac{3}{2}$. First, the absolute value of $f(z)$ gives using the backward triangle inequality, for $z \in \mathbf{c}$,

$$|f(z)| = \left|\frac{z}{z^2+1}\right| \le \frac{|z|}{|z|^2 - 1} \le \frac{2.5}{1.5^2 - 1} = 2.$$

Also $|z - 2\mathbf{i}| = \frac{1}{2}$. Thus integrating on the circle of radius R about the origin,

$$\left|\frac{d^{10} f}{d\,z^{10}}(2\mathbf{i})\right| = \left|\frac{10!}{2\pi\mathbf{i}}\oint_{\mathbf{c}} \frac{f(z)\,dz}{(z-2\mathbf{i})^{11}}\right| \le \frac{10!}{2\pi}\oint_{\mathbf{c}} \frac{|f(z)|\,|dz|}{|z-2\mathbf{i}|^{11}} \le \frac{2\cdot 10!}{2\pi\cdot(\frac{1}{2})^{11}}\oint_{\mathbf{c}} |dz| \le \frac{2\cdot 10!\cdot\pi}{2\pi\cdot(\frac{1}{2})^{11}} = 10!\cdot 2^{11}$$

(E.2.) *Suppose that* $f(z) = \displaystyle\sum_{k=-M}^{N} c_k\,z^k$ *be a rational function. Prove the Residue Theorem. That is, show that if* $\mathbf{c}$ *is a positively oriented simple closed contour in the plane enclosing a region* R *so that* $0 \in R$*, then*

$$\oint_{\mathbf{c}} f(z)\,dz = 2\pi\mathbf{i}\,c_{-1}.$$

Since 0 is an interior points of the enclosed region, there is a small radius $\varepsilon > 0$ so that the circle $\mathbf{c}'$ of radius ε about 0 also lies within the region R. Also, $f(z)$ is a finite sum of functions that are analytic in the punctured plane. $\mathbf{C} - \{0\}$. It follows that the integrals are equal. Furthermore, letting $z(t) = \varepsilon e^{\mathbf{i}\,t}$, so that $dz = \mathbf{i}\,\varepsilon e^{\mathbf{i}\,t}\,dt$ we find

$$\oint_{\mathbf{c}} f(z)\,dz = \oint_{\mathbf{c}'} f(z)\,dz = \sum_{k=-M}^{N} c_k \int_{\mathbf{c}'} z^k\,dz = \sum_{k=-M}^{N} c_k \int_0^{2\pi} \varepsilon^{k+1}\, e^{\mathbf{i}\,kt}\, \mathbf{i} e^{\mathbf{i}\,t}\,dt$$

$$= \sum_{k=-M}^{N} c_k\, \varepsilon^{k+1}\, \mathbf{i} \left\{ \begin{matrix} \left[\dfrac{e^{\mathbf{i}\,kt}}{(k+1)\mathbf{i}} \right]_0^{2\pi} = 0, & \text{if } k \neq -1, \\ \\ 2\pi, & \text{if } k = -1. \end{matrix} \right\} = 2\pi\,\mathbf{i}\,c_{-1}$$

(E.3.) *Suppose that* $\mathbf{c}$ *is a simple, closed, positively oriented contour in the palne. Show that*

$$\frac{1}{2\mathbf{i}} \oint_{\mathbf{c}} \bar{z}\,dz = \textit{Area of the region enclosed by } \mathbf{c}.$$

Let R be the region enclosed by $\mathbf{c}$. The function $\bar{z} = x - \mathbf{i}\,y$ is a smooth function, therefore Green's Theorem $\left(\oint_{\dot{R}} p\,dx + q\,dy = \iint_R (q_x - p_y)\,dx\,dy \right)$ applies. Writing as a line integral,

$$\frac{1}{2\mathbf{i}} \oint_{\mathbf{c}} \bar{z}\,dz = \frac{1}{2\mathbf{i}} \oint_{\mathbf{c}} (x - \mathbf{i}\,y)\,(dx + \mathbf{i}\,dy) = \frac{1}{2\mathbf{i}} \oint_{\mathbf{c}} (x - \mathbf{i}\,y)\,dx + (y + \mathbf{i}\,x)\,dy$$
$$= \frac{1}{2\mathbf{i}} \iint_R \left(\frac{\partial(y + \mathbf{i}\,x)}{\partial x} - \frac{\partial(x - \mathbf{i}\,y)}{\partial y} \right) dx\,dy$$
$$= \frac{1}{2\mathbf{i}} \iint_R (\mathbf{i} + \mathbf{i})\,dx\,dy = \text{Area}(R).$$

(E.4.) *Show that* $|\sinh x| \le |\cosh z| \le \cosh x$ *for all* $z = x + \mathbf{i}\,y \in \mathbf{C}$.

$$\cosh z = \frac{e^z + e^{-z}}{2} = \frac{e^x(\cos y + \mathbf{i}\,\sin y) + e^{-x}(\cos y - \mathbf{i}\,\sin y)}{2}$$
$$= \left(\frac{e^x + e^{-x}}{2} \right) \cos y + \mathbf{i}\,\left(\frac{e^x - e^{-x}}{2} \right) \sin y = \cosh x\,\cos y + \mathbf{i}\,\sinh x\,\sin y$$

Hence

$$\begin{aligned} |\cosh z|^2 &= \cosh^2 x\,\cos^2 y + \sinh^2 x\,\sin^2 y \\ &= (1 + \sinh^2 x)\,\cos^2 y + \sinh^2 x\,(1 - \cos^2 y) \\ &= \cos^2 y + \sinh^2 x \end{aligned}$$

Thus

$$\sinh^2 x \le \sinh^2 x + \cos^2 y = |\cosh z|^2 \le \sinh^2 x + 1 = \cosh^2 x$$

as desired.

(E.4.) *Suppose that* $\mathbf{c}$ *is the diamond-shaped simple closed contour consisting of straight line segments from* 5 *to* $1 + 4\mathbf{i}$ *to* -3 *to* $1 - 4\mathbf{i}$ *and back to* 5. *Find*

$$\oint_{\mathbf{c}} \frac{z\,dz}{(z^4 - 256)^2}$$

Factoring, we find $z^4 - 256 = (z^2 + 16)(z^2 - 16) = (z + 4\mathbf{i})(z - 4\mathbf{i})(z + 4)(z - 4)$. Thus the only singularity within $\mathbf{c}$ is $z = 4$. Hence

$$I = \oint_{\mathbf{c}} \frac{z\,dz}{(z^4 - 256)^2} = \oint_{\mathbf{c}} \frac{f(z)\,dz}{(z-4)^2} = 2\pi\,\mathbf{i}\,f'(4)$$

where

$$f(z) = \frac{z}{(z^2 + 16)^2 (z + 4)^2}$$

is analytic on and within $\mathbf{c}$. But

$$f'(z) = \frac{64 - 16z - 12z^2 - 5z^3}{(z^2 + 16)^3 (z + 4)^3}, \qquad \text{so} \qquad I = 2\pi\,\mathbf{i}\,f'(4) = 2\pi\,\mathbf{i}\,f'(4) = -(0.0001917\ldots)\mathbf{i}$$