Math 3160 § 1. **Second Midterm Exam** Name: Midterm Exam Solutions
Treibergs March 30, 2006

1. *Find all* $(-1)^{\mathbf{i}}$. *Find the principal value of* $(-1)^{\mathbf{i}}$.

$-1 = e^{\pi \mathbf{i}}$ so that $\log(-1) = \{\ln 1 + (\pi + 2\pi k)\mathbf{i} : k \in \mathbf{Z}\}$. Thus

$$(-1)^{\mathbf{i}} = \exp\left(\mathbf{i}\,\log(-1)\right) = \exp\left(\mathbf{i}\,\{(\pi + 2\pi k)\mathbf{i} : k \in \mathbf{Z}\}\right) = \{\exp\left(-\pi + 2\pi k\right) : k \in \mathbf{Z}\}.$$

The principal value is gotten by using the principal logarithm by taking $k = 0$. Thus P.V. $(-1)^{\mathbf{i}} = e^{-\pi}$.

2. *Let* $\mathbf{C}$ *be the rectangular contour consisting of line segments from* 2 *to* $2 + \mathbf{i}$ *to* $-2 + \mathbf{i}$ *to* -2 *and back to* 2. *Find*

$$I = \oint_C \frac{2z\,dz}{4z^2 + 1}$$

There is more than one way to skin a cat. Here is the solution I gave in 2006 based on the Cauchy Integral Formula.

Factoring the denominator we find $4z^2 + 1 = 2(2z + \mathbf{i})(z - \frac{1}{2}\mathbf{i})$ so that the poles of the function are at $\pm\frac{1}{2}\mathbf{i}$. Only the upper one is inside the box. Putting $f(z) = \dfrac{z}{2z + \mathbf{i}}$ we have by the Cauchy Integral Theorem,

$$I = \oint_{\mathbf{c}} \frac{f(z)\,dz}{z - \frac{1}{2}\mathbf{i}} = 2\pi\mathbf{i} f\left(\frac{1}{2}\mathbf{i}\right) = 2\pi\mathbf{i} \cdot \frac{\frac{1}{2}\mathbf{i}}{2 \cdot \frac{1}{2}\mathbf{i} + \mathbf{i}} = \frac{\pi\mathbf{i}}{2}.$$

Now here is a solution using the corollary to the Cauchy-Goursat Theorem instead (within the scope of our 2015 class!)

Notice that the denominator factors

$$4z^2 + 1 = (2z + i)(2z - i)$$

so that the integrand

$$f(z) = \frac{2z\,dz}{4z^2 + 1}$$

is singular at $z = \pm i/2$. Only $i/2$ is within C. Using partial fractions, we can write

$$f(z) = \frac{\frac{1}{2}}{2z + i} + \frac{\frac{1}{2}}{2z - i}$$

so that the integral equals

$$I = \frac{1}{2}\int_C \frac{dz}{2z + i} + \frac{1}{2}\int_C \frac{dz}{2z - i} = 0 + \frac{1}{2}\int_C \frac{dz}{2z - i}$$

because $\dfrac{1}{2z + i}$ is analytic inside and on C so by the Cauchy-Goursat Theorem its integral vanishes. Let C' be the circular contour $\left|z - \dfrac{i}{2}\right| = \dfrac{1}{3}$ with the counterclockwise orientation. Note that C' is interior to C and surrounds the singularity at $i/2$. Thus $\dfrac{1}{2z - i}$ is analytic on C and C' as well as in the annular region inside C and outside C'. It follows from the multiply connected Cauchy-Goursat Theorem, that the integrals over C and over C' are the same, thus

$$I = \frac{1}{2}\int_{C'} \frac{dz}{2z - i}.$$

The easiest way to finish is just to parameterize the circle $z(\theta) = \dfrac{1}{3}e^{i\theta} + \dfrac{i}{2}$ for $0 \le \theta \le 2\pi$ and compute

$$I = \frac{1}{2}\int_{C'} \frac{dz}{2z-i} = \frac{1}{2}\int_0^{2\pi} \frac{z'(\theta)\,d\theta}{2z(\theta)-i} = \frac{1}{2}\int_0^{2\pi} \frac{\frac{i}{3}e^{i\theta}\,d\theta}{\frac{2}{3}e^{i\theta}} = \frac{i}{4}\int_0^{2\pi} d\theta = \frac{\pi i}{2}.$$

A similar computation results if we don't do partial fractions. Also we could have spit the integral into an upper and lower contour and used antiderivatives involving logarithms instead of the direct computation around C'.

3. *Let* $\mathbf{C}$ *be the line segment from* 1 *to* $\mathbf{i}$. *Find*

$$J = \int_C \bar{z}^2\,dz$$

Because $\bar{z}^2$ is not analytic, we have to do the integral by hand. Parameterizing the linear path, $z(t) = (1-t) + t\,\mathbf{i}$ so that $dz = (-1+\mathbf{i})\,dt$. Also $\overline{z(t)} = (1-t) - t\,\mathbf{i}$ so that $\overline{z(t)}^2 = (1-t)^2 - t^2 - 2(1-t)t\,\mathbf{i} = 1 - 2t - 2(t-t^2)\,\mathbf{i}$. Thus

$$J = \int_0^1 \left[1 - 2t - 2(t-t^2)\mathbf{i}\right](-1+\mathbf{i})\,dt = (-1+\mathbf{i})\left[t - t^2 - \left(t^2 - \frac{2}{3}t^3\right)\mathbf{i}\right]_0^1 = \frac{1+\mathbf{i}}{3}$$

4. *Let* $\mathbf{C}$ *be the contour consisting of five line segments* 1 *to* $e^{\pi\mathbf{i}/3}$ *to* $e^{2\pi\mathbf{i}/3}$ *to* -1 *to* $e^{4\pi\mathbf{i}/3}$ *to* $e^{5\pi\mathbf{i}/3}$. *Find*

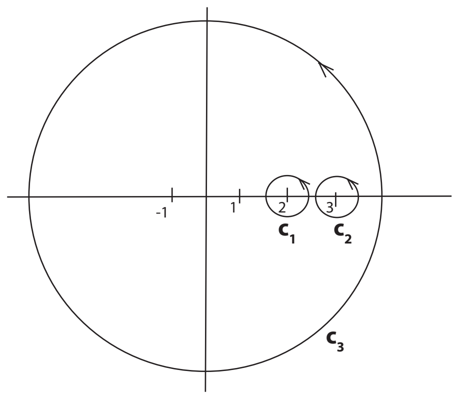
$$K = \int_C \frac{dz}{z}$$

The contour starts at $z_1 = 1$ and ends at $z_2 = e^{5\pi\mathbf{i}/3}$. Take the branch of logarithm defined for $z = re^{i\vartheta}$ by $F(z) = \log z = \ln r + \mathbf{i}\,\vartheta$ where $-\frac{\pi}{4} < \vartheta < \frac{7\pi}{4}$. This branch is analytic in a domain ($\mathbf{C}$ slit from origin in the $\vartheta = -\pi/4$ direction) containing the contour. Furthermore, it is the antiderivative there: $F'(z) = \frac{1}{z}$ in the slit domain. Thus

$$K = F(z_2) - F(z_1) = \mathbf{i}\,\frac{5\pi}{6} - \mathbf{i}\cdot 0 = \frac{5\pi\mathbf{i}}{6}.$$

5. *Suppose that the three closed contours are as in the diagram. Explain why*

$$\oint_{\mathbf{c}_3} \frac{\cos z\,dz}{z^2-5z+6} = \oint_{\mathbf{c}_1} \frac{\cos z\,dz}{z^2-5z+6} + \oint_{\mathbf{c}_2} \frac{\cos z\,dz}{z^2-5z+6}.$$



Let $f(z) = \dfrac{\cos z}{z^2-5z+6}$. Since $z^2-5z+6 = (z-2)(z-3)$ is zero only at $z_1 = 2$ or $z_2 = 3$ and since $\cos z$ is entire, the integrand $f(z)$ is analytic everywhere except at the two points $z_1,\ z_2$ inside the simple, closed, positively oriented contours $\mathbf{c}_1$ $\mathbf{c}_2$, resp. Since all the contours are disjoint, and that $\mathbf{c}_3$ is a closed, simple, positively oriented contour which surrounds the other two, then $f(z)$ is analytic on all of the $\mathbf{c}_i$ and in the region R which is inside of $\mathbf{c}_3$ and outside both $\mathbf{c}_1$ and $\mathbf{c}_2$. The conclusion follows from the extension of the Cauchy Goursat Theorem to multiply connected domains. The argument goes as follows. Cut R into two pieces along the x-axis and call the line segments on the x-axis in R from $\mathbf{c}_3$ to $\mathbf{c}_1$ by α, from $\mathbf{c}_1$ to $\mathbf{c}_2$ by β and from $\mathbf{c}_2$ to $\mathbf{c}_3$ by γ. Now let the upper portion of R be denoted R^+ and the lower R^-. Let the contours also be split into upper and lower portions, *e.g.*, $\mathbf{c}_2 = \mathbf{c}_2^+ + \mathbf{c}_2^-$. Now the usual Cauchy-Goursat Theorem applies to R^+ because f is analytic in it and on its boundary $\alpha - \mathbf{c}_1^+ + \beta - \mathbf{c}_2^+ + \gamma + \mathbf{c}_3^+$. Similarly, f is analytic on R^- and the boundary of R^- is $-\gamma - \mathbf{c}_2^- - \beta - \mathbf{c}_1^- - \alpha + \mathbf{c}_3^-$. The Cauchy Goursat integrals around ∂R^+ and ∂R^- give

$$0 = \int_{\alpha-\mathbf{c}_1^+ +\beta-\mathbf{c}_2^+ +\gamma+\mathbf{c}_3^+} f(z)\,dz, \qquad 0 = \int_{-\gamma-\mathbf{c}_2^- -\beta-\mathbf{c}_1^- -\alpha+\mathbf{c}_3^-} f(z)\,dz$$

Adding,

$$\begin{aligned} 0 &= \int_{\alpha-\mathbf{c}_1^+ +\beta-\mathbf{c}_2^+ +\gamma+\mathbf{c}_3^+ -\gamma-\mathbf{c}_2^- -\beta-\mathbf{c}_1^- -\alpha+\mathbf{c}_3^-} f(z)\,dz = \int_{-\mathbf{c}_1^+ -\mathbf{c}_2^+ +\mathbf{c}_3^+ -\mathbf{c}_2^- -\mathbf{c}_1^- +\mathbf{c}_3^-} f(z)\,dz \\ &= -\int_{\mathbf{c}_1} f(z)\,dz - \int_{\mathbf{c}_2} f(z)\,dz + \int_{\mathbf{c}_3} f(z)\,dz \end{aligned}$$

as desired.