Math 3160 § 1. **First Midterm Exam** Name: Sample
Treibergs $\sigma\tau$ February 16, 2006

Midterm exam given Fri., Jan 30, 1998.

(M.1.) *Let* $z = 2e^{\frac{\pi i}{3}}$. *Find* z^2, $\bar{z}$, $\dfrac{1}{z}$, $\arg z$, $\operatorname{Arg} z$, $\Re\mathrm{e}\, z$ *and* $|z|$.

(M.2.) *Using the fact that* $w = \cot z = \dfrac{ie^{iz} + ie^{-iz}}{e^{iz} - e^{-iz}}$, *find* $\cot^{-1} w$.

[Hint: Let $p = e^{iz}$ *and solve for* p *in terms of* w. *Then* $iz = \log p$.*]*

(M.3.) *Find all possible* $(1+i)^{\frac{1}{6}}$

(M.4.) *Suppose that* $f(z)$ *is an analytic function for all* $z \in \mathbf{C}$ *so that* $\Re\mathrm{e}\, f(z) + \Im\mathrm{m}\, f(z) = 1$ *for all* z. *Show that* $f(z)$ *is constant.*

(M.5.) *Let* $g(z) = \bar{z}(z+i)$ *Find all possible* $z_0 \in \mathbf{C}$ *where the complex limit exists:*

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}.$$

Solutions of the midterm.

(M.1.) $z = 2e^{\frac{\pi i}{3}}$ so $z^2 = 4e^{\frac{2\pi i}{3}} = 2^2(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) = 4(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -2 + 2i\sqrt{3}$. $\bar{z} = 2e^{-\frac{\pi i}{3}} = 2(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}) = 2(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 1 - \sqrt{3}i$. $\frac{1}{z} = \frac{1}{2}e^{-\frac{\pi i}{3}} = \frac{1}{2}(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = \frac{1}{4} - i\frac{\sqrt{3}}{4}$. $\arg z = \{\frac{\pi}{3} + 2\pi k : k \in \mathbf{Z}\}$. $\theta = \operatorname{Arg} z = \frac{\pi}{3}$ because $-\pi < \theta \le \pi$. $|z| = 2$.

(M.2.) Let $p = e^{iz}$. Then

$$w = \cot z = \frac{ie^{iz} + ie^{-iz}}{e^{iz} - e^{-iz}} = \frac{i\left(p + \frac{1}{p}\right)}{p - \frac{1}{p}} = \frac{i(p^2+1)}{p^2 - 1}.$$

Thus, solving for p^2, $w(p^2 - 1) = i(p^2 + 1)$ so $p^2(w - i) = w + i$ so $p^2 = \dfrac{w+i}{w-i}$ so $p = \left(\dfrac{w+i}{w-i}\right)^{\frac{1}{2}}$. Finally,

$$z = \cot^{-1} w = -i \log p = \frac{i}{2} \log\left(\frac{w-i}{w+i}\right).$$

(M.3.) $z = 1 + i = \sqrt{2}e^{\frac{\pi i}{4}}$ so $z^{\frac{1}{6}} = \sqrt[12]{2}\left(e^{\frac{\pi i}{4}}\right)^{\frac{1}{6}} = \left\{\sqrt[12]{2} \exp\left(\frac{\pi i}{24} + \frac{\pi k i}{3}\right) : k \in \mathbf{Z}\right\}$.

(M.4.) $f(x+iy) = u(x,y) + iv(x,y)$ is analytic on $\mathbf{C}$ so that the Cauchy-Riemann equations hold: $u_x = v_y$ and $u_y = -v_x$. But we are told that equation also holds $u + v = 1$. Differentiating with respect to x and then to y gives $u_x + v_x = 0$ and $u_y + v_y = 0$. Plugging in the CR-Equations, $v_y + v_x = 0$ and $-v_x + v_y = 0$. Subtracting the last two equations gives $2v_x = 0$ so using the differentiated equation, $0 = u_x + v - x = u_x + 0$ implies $u_x = 0$. Hence, the complex derivative $f'(z) = u_x + iv_x = 0$ vanishes for all z in the connected domain $\mathbf{C}$. But we showed that a function that is analytic in a domain and with zero complex derivative has to be constant.

(M.5.) We show that the complex derivative exists and equals $f'(-i) = -i$ when $z_0 = -i$, but exists for no other z_0. Writing $h = z - z_0 (\neq 0)$, the difference quotient becomes

$$\begin{aligned} Q = \frac{f(z_0+h) - f(z_0)}{h} &= \frac{\overline{(z_0+h)}(z_0+h+i) - \bar{z}_0(z_0+i)}{h} \\ &= \frac{\bar{z}_0(z_0+h+i) + \bar{h}(z_0+h+i) - \bar{z}_0(z_0+i)}{h} \\ &= \frac{\bar{z}_0 h + \bar{h}(z_0+h+i)}{h} = z_0 + \frac{\bar{h}}{h}(z_0+i) + \bar{h}. \end{aligned}$$

If $z_0+i \ne 0$ then ther is no complex limit: The approach $h = x+0i \to 0$ implies $Q \to z_0+(z_0+i)+0$ whereas along the vertical approach $h = 0 + yi \to 0$ then $Q \to z_0 - (z_0 + i) + 0$. The two limits disagree so there is no limit. However,if $z_0 + i = 0$ then $Q = z_0 + \bar{h} \to z_0$ no matter how $h \to 0$.

Some extra practice problems.

(E.1.) *Find the locus of points $z \in \mathbf{C}$ that satisfy $|z + 1| = |z - 2i|$.*

Rewriting, the equation is $|z-(-1)| = |z-(2i)|$. Thus the locus is those z that are equidistant from the points -1 and $2i$. This is the bisecting line between those two points, or $y = -\frac{1}{2}x + \frac{3}{4}$. The line is perpendicular to the segment from (-1,0) to (0,2) of slope 2 so the line has slope $-\frac{1}{2}$. The constant is chosen to make sure that line passes through the midpoint $-\frac{1}{2} + i$.

(E.2.) *Show that $|2 + z^2 + \Im\mathrm{m}\, z| \le 22$ if $|z| \le 4$.*

Triangle inequality. Assuming $|z| \le 4$ gives

$$|2 + z^2 + \Im\mathrm{m}\, z| \le |2| + |z^2| + |\,\Im\mathrm{m}\, z| \le 2 + |z|^2 + |z| \le 2 + 4^2 + 4 = 22.$$

(E.3.) *Show for all real θ,*

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = \frac{\sin\frac{9}{2}\theta}{2\sin\frac{1}{2}\theta}.$$

Using the formula for the geometric series, $1 + z + z^2 + z^3 + z^4 = \frac{1-z^5}{1-z}$, with $z = e^{i\theta}$ and taking the real part,

$$\Re\mathrm{e}\left(1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} + e^{4i\theta}\right) = \Re\mathrm{e}\left(\frac{1-e^{5i\theta}}{1-e^{i\theta}}\right) = \Re\mathrm{e}\left(\frac{(1-e^{5i\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}\right)$$
$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = \Re\mathrm{e}\left(\frac{1-e^{5i\theta}-e^{-i\theta}+e^{4i\theta}}{2-e^{i\theta}-e^{-i\theta}}\right) = \frac{\Re\mathrm{e}(1-e^{-i\theta}-e^{5i\theta}+e^{4i\theta})}{2-2\cos\theta}$$
$$= \frac{1-\cos\theta-\cos 5\theta+\cos 4\theta}{2-2\cos\theta} = \frac{1}{2} + \frac{\cos 4\theta - \cos 5\theta}{2(1-\cos\theta)} = \frac{1}{2} + \frac{2\sin(\frac{9}{2}\theta)\sin(\frac{1}{2}\theta)}{4\sin^2\frac{\theta}{2}} = \frac{1}{2} + \frac{\sin(\frac{9}{2}\theta)}{2\sin\frac{\theta}{2}}$$

where we used the trig identity $1 - \cos\theta = 1 - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$ and $\cos 4\theta - \cos 5\theta = \cos(\frac{9}{2}\theta - \frac{1}{2}\theta) - \cos(\frac{9}{2}\theta + \frac{1}{2}\theta) = 2\sin(\frac{9}{2}\theta)\sin(\frac{1}{2}\theta)$. This is the desired result.

(E.4.) *For the set S, find the interior points, the boundary points, the accumulation points. Is S connected? Is S a domain?*

$$S = \{z \in \mathbf{C} :\ 0 < |z| < 1\ \} \cup \{z \in \mathbf{C} :\ |z| = 1 \ \ \textit{and}\ \ \Re\mathrm{e}\, z \ge 0\ \} \cup \{2i\}$$

The set S consists of the unit punctured disk about the origin, with half of the unit circle included with an isolated point at $2i$. The interior points are the punctured disk $\operatorname{int} S = \{z \in \mathbf{C} : 0 < |z| < 1\}$. The boundary $\dot{S} = S = \{z \in \mathbf{C} : |z| = 1\} \cup \{0\} \cup \{2i\}$. The accumulation points $S' = \{z \in \mathbf{C} : |z| \le 1\}$. S is not connected because there is no zig-zag path connecting the isolated point $\{2i\}$ to the rest of the set. S is not connected so not a domain. (S isn't open either.)

(E.4.) *Let $f(z) = e^z$. Find $f(S)$, where $S = \{z \in \mathbf{C} :\ \Re\mathrm{e}\, z \le 0\ \textit{and}\ \Im\mathrm{m}\, z > 0\ \}$.*

S is the second quadrant. As the exponential function is $w = f(z) = e^x(\cos y + i\sin\theta)$. Since $x + iy \in S$ implies $-\infty < x \le 0$ so $0 < r = e^x \le 1$. On the other hand $x + iy \in S$ implies $y > 0$ which includes infinitely many 2π-periods. Thus $\cos x + i\sin x$ takes on all values in the unit circle. Thus $f(S) = \{re^{i\theta} :\ 0 < r \le 1 \text{ and } \theta \ge 0\ \}$ is the closed unit disk minus the origin.

(E.5.) *What is the domain of* $f(z) = \dfrac{1}{z^2+2z+5}$*?*

The domain is all $z \in \mathbf{C}$ so that the denominator is nonzero. But the denominator is zero when $z^2+4z+5=0$. Using the quadratic formula

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4\cdot 1\cdot 5}}{2\cdot 1} = -2 \pm i.$$

Thus the domain is $\{z \in \mathbf{C} : z \neq -2 \pm i\}$.

(E.6.) *Assuming everything about finite limits (* $L = \lim_{z\to z_0} f(z)$ *when* z_0 *and* L *are finite), find and explain* $L = \lim\limits_{z\to\infty} \left(z^2+z^3\right)$.

The limit exists and $L = \infty$. The limit $\lim\limits_{z\to\infty} f(z) = \infty$ means (by replacing both $z = 1/w$ and $f(z)$ by their reciprocals)

$$0 = \lim_{w\to 0} \frac{1}{f\left(\frac{1}{w}\right)}.$$

To check this for the present case, $f(z) = z^2+z^3$ so that

$$\lim_{w\to 0} \frac{1}{f\left(\frac{1}{w}\right)} = \lim_{w\to 0} \frac{1}{\frac{1}{w^2}+\frac{1}{w^3}} = \lim_{w\to 0} \frac{w^3}{w+1} = \frac{\lim_{w\to 0} w^3}{\lim_{w\to 0} w + \lim_{w\to 0} 1} = \frac{0^3}{0+1} = 0.$$

(E.7.) *Does* $f(x+iy) = \dfrac{x+iy}{x+2iy}$ *have a complex limit as* $x+iy \to 0$*? Why?*

There is no complex limit. Along the horizontal approach $z = x+0i \to 0$ then $f(x+0i) = \dfrac{x}{x} = 1 \to 1$. On the other hand, along the vertical approach $z = 0+iy \to 0$ then $f(0+iy) = \dfrac{iy}{2iy} = \dfrac{1}{2} \to \dfrac{1}{2}$. As both approaches have inconsistent limiting values, there is no complex limit.

(E.8.) *Is* $f(x+iy) = x+2yi$ *an entire function? Why?*
The Cauchy Riemann equations fail at all points, so f does not have a complex derivative anywhere. Since $u = x$ and $v = 2y$ we have $u_x = 1$ and $v_y = 2$. Thus the CR equation $u_x = v_y$ holds nowhere.

(E.9.) *Does* $f(x+iy) = (y\cos x + x\sin x)e^{-y} + i(y\sin x - x\cos x)e^{-y}$ *have a complex derivative at* x_0+iy_0*? Why? If it is defferentiable, what is* f'*?*

We claim that this f has a complex derivative at all points. We are given $u(x,y) = (y\cos x + x\sin x)e^{-y}$ and $v(x,y) = (y\sin x - x\cos x)e^{-y}$. The partial derivatives are $u_x = (-y\sin x + \sin x + x\cos x)e^{-y}$, $u_y = (\cos x - y\cos x - x\sin x)e^{-y}$, $v_x = (y\cos x - \cos x + x\sin x)e^{-y}$ and $v_y = (\sin x - y\sin x + x\cos x)e^{-y}$. First, these are combinations of exponential, sine and cosine functions, and therefore are continuous at all points. Furthermore, the Cauchy-Riemann Equations hold: $u_x = v_y$ and $u_y = -v_x$. When the partial derivatives are continuous and the Cauchy-Riemann equations hold, then the complex derivative exists and it equals $f'(z) = u_x + iv_x = (-y\sin x + \sin x + x\cos x)e^{-y} + i(y\cos x - \cos x + x\sin x)e^{-y}$

(E.10.) *Let* $u(x,y) = x^3 - 3xy^2$*. Show that* u *is harmonic. Find its harmonic conjugate.*

$u_{xx} = 6x$. $u_{yy} = -6x$. Thus u satisfies $u_{xx}+u_{yy} = 0$ so is harmonic. To find the harmonic conjugate we solve the CR equations for $v(x,y)$.

$$3x^2 - 3y^2 = u_x = v_y \qquad \text{and} \qquad 6xy = -u_y = v_x.$$

Partially integrating the second equation, $v(x,y) = 3x^2y + k(y)$. Differentiating and using the first equation, $3x^2-3y^2 = 3x^2 + k'(y)$ therefore $k(y) = -y^3 + c$. Finally, the harmonic conjugate is $v(x,y) = 3x^3x^2y - y^3 + c$.

(E.11.) *Suppose* $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ *is analytic function on a domain* $D \subset \mathbf{C}$. *Show that* u *is harmonic.*

Being analytic, the higher partial derivatives all exist and are continuous. Also the Cauchy Riemann equations in polar coordinates hold $ru_r = v_\theta$ and $u_\theta = -rv_r$. Differentiating the first with respect to θ and multiplying by r gives $ru_r + r^2 u_{rr} = rv_{\theta r}$. Differentiating the second with respect to θ gives $u_{\theta\theta} = -rv_{r\theta}$. Equating the equal cross partials, $r^2 u_{rr} + ru_r + u_{\theta\theta} = 0$, which is the equation for a harmonic function in polar coordinates. Similarly differentiating the first equation with respect to θ yields $ru_{r\theta} = v_{\theta\theta}$. Differentiating the second with respect to r and multiplying by r gives $ru_{\theta r} = -rv_r - r^2 v_{rr}$. Equating the cross partials, we get the harmonic equation for the other function $0 = v_{\theta\theta} + rv_r + r^2 v_{rr}$.

(E.12.) *Suppose* f *is analytic function for* $\Re\mathrm{e}\, z > 0$. *Suppose that* $\Im\mathrm{m}\, f(x + i0) = 0$ *for all* $x > 0$. *If* $f(1+i) = \frac{\pi}{4} i$, *what is* $f(1-i)$? *Why?*

The domain has $z \to \bar{z}$ symmetry. Since the function is real on the x-axis, the reflection principle holds for the analytic function, or $\overline{f(z)} = f(\bar{z})$. But we are given f for $z = 1 + i$. It follows from the reflection principle $f(1-i) = f(\bar{z}) = \overline{f(z)} = \overline{f(1+i)} = \overline{\frac{\pi}{4} i} = -\frac{\pi}{4} i$.