Math 2270 § 1. **Third Midterm Exam** Name: Solutions
Treibergs $\sigma\tau$ November 14, 2018

1. *Let* $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix}$. *Find the characteristic polynomial of A. Find the eigenvalues. For each eigenvalue, find all eigenvectors.*

The characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & -2 \\ 1 & -1 & -\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(-\lambda) - 2(2-\lambda) = (2-\lambda)(\lambda^2-\lambda-2) = (2-\lambda)^2(1-\lambda)$$

so the eigenvalues are $\lambda_1 = -1$ with multiplicity one and $\lambda_2 = 2$ with multiplicity two. Solving for eigenvectors corresponding to $\lambda_1 = -1$ we have the eigenspace

$$\mathbf{0} = (A - \lambda_1 I)\mathbf{x}_1 = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \text{so} \qquad \mathcal{E}_{-1} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Solving for eigenvectors corresponding to $\lambda_2 = 2$ we have the eigenspace

$$\mathbf{0} = (A - \lambda_2 I)[\mathbf{x}_2, \mathbf{x}_3] = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & -2 \\ 1 & -1 & -2 \end{pmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{so} \qquad \mathcal{E}_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2. *Let* $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. *Show that the vectors* $\mathbf{a}_1$, $\mathbf{a}_2$ *and* $\mathbf{a}_3$ *are linearly independent. Find a vector* $\mathbf{a}_4$ *so that* $\mathcal{B} = \{\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3,\ \mathbf{a}_4\}$ *is a basis for* $\mathbf{R}^4$. *Explain why your choice works.*

Just about any vector will work, so we try $\mathbf{a}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Putting the vectors as columns of $A$ and row reducing,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There are no free variables, so the four columns of $A$ are independent, thefore the first three vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are linearly independent. Since the four vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$ are linearly independent in the four dimensional space $\mathbb{R}^4$, they form a basis.

3. *Let* $\mathbb{V} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{R} \right\}$ *be the vector space of* $2 \times 2$ *real matrices with the usual matrix addition and scalar multiplication. State four of the ten axioms satisfied by* $\mathbb{V}$ *to make it a vector space. For the* $1 \times 2$ *matrix* $B = \begin{pmatrix} 2 & 3 \end{pmatrix}$ *let* $\mathbb{W} = \left\{ X \in \mathbb{V} : BX = \begin{pmatrix} 0 & 0 \end{pmatrix} \right\}$ *be the subset of matrices satisfying the matrix equation. Show that* $\mathbb{W}$ *is a vector subspace of* $\mathbb{V}$ *by verifying that* $\mathbb{W}$ *satisfies the conditions to be a vector subspace. Find a basis for the vector subspace* $\mathbb{W}$.

You can state any four, but the first four are

- For every $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ the sum $\mathbf{x} + \mathbf{y}$ is in $\mathbb{V}$.
- For all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ there holds $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
- For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{V}$ there holds $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
- There is a zero vector $\mathbf{0}$ in $\mathbb{V}$ such that for every $\mathbf{x}$ in $\mathbf{V}$ there holds $\mathbf{x} + \mathbf{0} = \mathbf{x}$.

See *p.* 192 of the text for the complete list.

To show that $\mathbb{W}$ is a vector subspace of $\mathbb{V}$ we must show that it contains zero, that it is closed under vector addition and that it is closed under scalar multiplication.

- The zero of $\mathbb{V}$ is the matrix $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It satisfies the condition
$$B\mathbf{0} = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$
which is the condition to be in $\mathbb{W}$, so $\mathbf{0} \in \mathbb{W}$.
- Let $\mathbf{x}$ and $\mathbf{y}$ be any matrices in $\mathbb{W}$, hence they satisfy $B\mathbf{x} = B\mathbf{y} = \begin{pmatrix} 0 & 0 \end{pmatrix}$. Thus $B(\mathbf{x} + \mathbf{y}) = \begin{pmatrix} 0 & 0 \end{pmatrix}$ which is the condition to be in $\mathbb{W}$, so $\mathbf{x} + \mathbf{y} \in \mathbb{W}$.
- Let $c$ be any real number and $\mathbf{x}$ be any matrix in $\mathbb{W}$ so it satisfires $B\mathbf{x} = \begin{pmatrix} 0 & 0 \end{pmatrix}$. Thus $B(c\mathbf{x}) = cB\mathbf{x} = c \begin{pmatrix} 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$ which is the condition to be in $\mathbb{W}$, so $c\mathbf{x} \in \mathbb{W}$.

The matrices $\mathbf{x} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{12} \end{pmatrix}$ in $\mathbb{W}$ satisfy

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{aligned} 2x_{11} + 3x_{21} &= 0 \\ 2x_{12} + 3x_{22} &= 0 \end{aligned}$$

Thus $x_{21}$ and $x_{22}$ are free, $x_{11} = -\frac{3}{3}x_{21}$ and $x_{12} = -\frac{3}{2}x_{22}$. It follows that the solutions are

$$\mathbb{W} = \left\{ \begin{pmatrix} -\frac{3}{2}x_{21} & -\frac{3}{2}x_{22} \\ x_{21} & x_{22} \end{pmatrix} : x_{21}, x_{22} \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} -\frac{3}{2} & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{3}{2} \\ 0 & 1 \end{pmatrix} \right\} = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$$

The matrices $\mathbf{b}_1$ and $\mathbf{b}_2$ are not multiples of each other, thus are linearly independent so form a basis of $\mathbb{W}$.

4. (a) *Let* $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. *Suppose* $A\mathbf{x} = \mathbf{b}$. *Find* $x_3$ *using Cramer's rule. (Other methods will receive no points.)*

$$x_3 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}} = \frac{4 + 3 + 0 - 0 - 0 - 2}{0 + 1 + 0 - 0 - 0 - 2} = \frac{5}{-1} = -5.$$

(b) *Suppose $A$ is $m \times n$ and $B$ is $n \times p$. By comparing column space of $AB$ to the column space of $A$, show that* $\operatorname{rank} AB \le \operatorname{rank} A$.

The column space of $A$ is

$$\operatorname{Col} A = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$$

and the column space of $AB$ is

$$\operatorname{Col} AB = \{AB\mathbf{y} : \mathbf{y} \in \mathbb{R}^p\} = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x} = B\mathbf{y} \text{ for some } \mathbf{y} \in \mathbb{R}^p.\}$$

Since the second set is more restricted, $\operatorname{Col} AB \subset \operatorname{Col} A$ so

$$\operatorname{rank} AB = \dim \operatorname{Col} AB \le \dim \operatorname{Col} A = \operatorname{rank} A.$$

5. *Let $\mathbb{P}_2 = \left\{a_0 + a_1 t + a_2 t^2 : a_1, a_2, a_3 \in \mathbf{R}\right\}$ be the vector space of polynomials of degree at most two. Let $\mathcal{B} = \left\{1+t,\ t+t^2,\ 1+t^2\right\}$, $\mathcal{C} = \left\{1,\ 1+t,\ 1+t+t^2\right\}$ be bases for $\mathbb{P}_2$. Recall that $[f(t)]_{\mathcal{B}}$ denotes the coordinates of $f \in \mathbb{P}_2$ in the $\mathcal{B}$ basis. Find $[6t+2t^2]_{\mathcal{B}}$ and $[6t+2t^2]_{\mathcal{C}}$. Find the matrix for changing coordinates from the $\mathcal{B}$ basis to the $\mathcal{C}$ basis $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Check that your matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ transforms $[6t+2t^2]_{\mathcal{B}}$ to $[6t+2t^2]_{\mathcal{C}}$ by multiplying coordinates you found in (a) by the matrix from (b).*

If $[6t+2t^2]_{\mathcal{B}} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ then

$$6t + 2t^2 = k_1(1+t) + k_2(t+t^2) + k_3(1+t^2).$$

Equating constants, $t$'s and $t^2$'s

$$\begin{aligned} k_1 + \phantom{k_2} \quad k_3 &= 0 \\ k_1 + k_2 \quad\quad &= 6 \\ k_2 + k_3 &= 2 \end{aligned} \quad \to \quad \begin{aligned} k_1 + \quad k_3 &= 0 \\ k_2 - k_3 &= 6 \\ k_2 + k_3 &= 2 \end{aligned} \quad \to \quad \begin{aligned} k_1 + \quad k_3 &= 0 \\ k_2 - k_3 &= 6 \\ 2k_2 \quad\quad &= 8 \end{aligned}$$

so $k_2 = 4$, $k_3 = -2$ and $k_1 = 2$. Thus $[6t+2t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$. Also, by inspection,

$$6t + 2t^2 = -6(1) + 4(1+t) + 2(1+t+t^2).$$

so $[6t+2t^2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$. If $[f]_{\mathcal{B}} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ then $f = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + k_3\mathbf{b}_3$. Taking the coordinate operator

$$[f]_{\mathcal{C}} = k_1[\mathbf{b}_1]_{\mathcal{C}} + k_2[\mathbf{b}_2]_{\mathcal{C}} + k_3[\mathbf{b}_3]_{\mathcal{C}} = \begin{pmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_1]_{\mathcal{C}} \end{pmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [f]_{\mathcal{B}}.$$

Computing by inspection

$$\begin{aligned} \mathbf{b}_1 &= 1 + t = 0 \cdot 1 + 1(1+t) + 0(1+t+t^2) \\ \mathbf{b}_2 &= t + t^2 = (-1) \cdot 1 + 0(1+t) + 1(1+t+t^2) \\ \mathbf{b}_3 &= 1 + t^2 = (1) - (1+t) + (1+t+t^2) \end{aligned}$$

so $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $[\mathbf{b}_3]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Hence $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$.

Checking,

$$[6t + 2t^2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix} \stackrel{?}{=} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [6t + 2t^2]_{\mathcal{B}}$$

we see that the matrix product equals the $\mathcal{C}$ coordinates.