Math 2270 § 1. **First Midterm Exam** Name: Solutions
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1. *Find the general solution to the linear system.*

$$\begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 3 \\ 2x_1 + x_2 - 2x_3 - x_4 &= 0 \\ x_1 + 5x_2 - 10x_3 - 5x_4 &= -9 \end{aligned}$$

Perform row operations on the augmented matrix.

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 3 \\ 2 & 1 & -2 & -1 & 0 \\ 1 & 5 & -10 & -5 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 & 3 \\ 0 & 3 & -6 & -3 & -6 \\ 0 & 6 & -12 & -6 & -12 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_3 - R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & -1 & 2 & 1 & 3 \\ 0 & 1 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } \frac{1}{3}R_2 \\ \text{Replace } R_3 \text{ by } R_3 - 2R_2 \end{matrix}$$

The last equation equates zero to zero so the system is consistnt. x_3 and x_4 are free variables and can be set to any real value. Solving we find

$$\begin{aligned} x_2 &= -2 + 2x_3 + x_4 \\ x_1 &= 3 + x_2 - 2x_3 - x_4 = 3 + (-2 + 2x_3 + x_4) - 2x_3 - x_4 = 1. \end{aligned}$$

Thus the set of solutions is

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -2 + 2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} : \text{where } x_3 \text{ and } x_4 \text{ are any real numbers.} \right\}$$

2. *Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be given by $T(\mathbf{x}) = A\mathbf{x}$ where*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & -3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} g \\ h \\ k \end{bmatrix}.$$

Is $\mathbf{b}$ *in the range of T? Explain why or why not. For which values of g, h and k is* $\mathbf{c}$ *in the range of T?*

Do row operations on the doubly augmented matrix $[A|\mathbf{b}|\mathbf{c}]$.

$$\begin{pmatrix} 1 & 2 & -1 & 3 & g \\ 1 & 1 & 1 & 1 & h \\ 1 & 3 & -3 & 0 & k \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 3 & g \\ 0 & -1 & 2 & -2 & h-g \\ 0 & 1 & -2 & -3 & k-g \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } R_2 - R_1 \\ \text{Replace } R_3 \text{ by } R_3 - R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & -1 & 3 & g \\ 0 & 1 & -2 & 2 & g-h \\ 0 & 0 & 0 & -5 & h-2g+k \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } -R_2 \\ \text{Replace } R_3 \text{ by } R_3 + R_2 \end{matrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is inconsistent because the RREF has zeros equal to -5 in the $\mathbf{b}$ column. Thus $\mathbf{b}$ is not the image of any $\mathbf{x}$. Thus $\boxed{\mathbf{b} \text{ is not in the range of } T(\mathbf{x}) = A\mathbf{x}}$.

$\mathbf{c}$ is in the range of $T(\mathbf{x}) = A\mathbf{x}$ if the equation $A\mathbf{x} = \mathbf{c}$ is consistent. Thus the last entry in the $\mathbf{c}$ column must be zero, or $\boxed{\mathbf{c} \text{ is in the range of } T(\mathbf{x}) = A\mathbf{x} \text{ if and only if } h - 2g + k = 0}$.

3. *A rotation $R : \mathbf{R}^2 \to \mathbf{R}^2$ is given by*

$$R\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} .6x_1 - .8x_2 \\ .8x_1 + .6x_2 \end{bmatrix}.$$

Define: $R(\mathbf{x})$ is a linear transformation*. Is $R(\mathbf{x})$ a linear transformation? Explain why or why not. Define: The map $R(\mathbf{x})$ is* onto*. Is $R(\mathbf{x})$ onto? Explain why or why not. Define: The map $R(\mathbf{x})$ is* one-to-one*. Is $R(\mathbf{x})$ one-to-one? Explain why or why not.*

A mapping $R : \mathbf{R}^2 \to \mathbf{R}^2$ is *linear* if for every choice of $\mathbf{v}, \mathbf{w} \in \mathbf{R}^2$ and every $c \in \mathbf{R}$ we have (1) $R(\mathbf{v} + \mathbf{w}) = R(\mathbf{v}) + R(\mathbf{w})$ and (2) $R(c\mathbf{v}) = cR(\mathbf{v})$. Observe that rotation is a matrix multiplication $R(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{pmatrix} .6 & -.8 \\ .8 & .6 \end{pmatrix}$. From matrix multiplication properties we have

$$\begin{aligned} R(\mathbf{v} + \mathbf{w}) = A(\mathbf{v} + \mathbf{w}) &= (A\mathbf{v}) + (A\mathbf{w}) = R(\mathbf{v}) + R(\mathbf{w}) \\ R(c\mathbf{v}) = A(c\mathbf{v}) &= c(A\mathbf{v}) = cR(\mathbf{v}). \end{aligned}$$

thus $R(\mathbf{x})$ is linear.

A mapping $R : \mathbf{R}^2 \to \mathbf{R}^2$ is *onto* if for every choice of $\mathbf{b} \in \mathbf{R}^2$ there exists at least one $\mathbf{x}$ such that $R(\mathbf{x}) = \mathbf{b}$. Writing the augmented matrix $[A, \mathbf{b}]$ for arbitrary $\mathbf{b}$ and doing row operations

$$\begin{aligned} \begin{pmatrix} .6 & -.8 & b_1 \\ .8 & .6 & b_2 \end{pmatrix} &\to \begin{pmatrix} 3 & -4 & 5b_1 \\ 4 & 3 & 5b_2 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_1 \text{ by } 5R_1 \\ \text{Replace } R_2 \text{ by } 5R_2 \end{matrix} \\ &\to \begin{pmatrix} 3 & -4 & 5b_1 \\ 0 & \frac{25}{3} & 5b_2 - \frac{20}{3}b_1 \end{pmatrix} \quad \text{Replace } R_2 \text{ by } R_2 - \tfrac{4}{3}R_1 \end{aligned}$$

Thus there is a pivot in each row of the reduced echelon form, thus the system may be solved for x_1 and x_2. In other words, $\boxed{R(\mathbf{x}) \text{ is onto.}}$.

A mapping $R : \mathbf{R}^2 \to \mathbf{R}^2$ is *one-to-one* if for every choice of $\mathbf{b} \in \mathbf{R}^2$ there exists at most one $\mathbf{x}$ such that $R(\mathbf{x}) = \mathbf{b}$. In the reduced echelon form above there are no free variables. Thus whenever $\mathbf{b}$ is such that the system $A\mathbf{x} = \mathbf{b}$ is consistent (we showed above that it is always consistent), then exactly one solution is possible. If it weren't consistent, then no solutions exists. In other words, $\boxed{R(\mathbf{x}) \text{ is one-to-one.}}$.

4. *Define: the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ is* linearly independent*. Determine whether the set $\mathcal{S}$ is linearly independent. Explain.*

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -2 \\ -1 \end{bmatrix} \right\}$$

The set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ is *linearly independent* if whenever we have the equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = 0$$

then $x_1 = x_2 = \cdots = x_p = 0$ is the only possible set of coefficients. Putting the vectors in as columns of A and row reducing the equation $A\mathbf{x} = 0$,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } R_2 - R_1 \\ \text{Replace } R_3 \text{ by } R_3 - R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } -R_2 \\ \text{Replace } R_3 \text{ by } R_3 + 2R_2 \\ \text{Replace } R_4 \text{ by } R_4 + R_2 \end{matrix}$$

The homogeneous system $A\mathbf{x} = 0$ has a free variable. Setting $x_3 = 1$ we find by back substitution that $x_2 = 1$ and $x_1 = -1$. Thus the vectors $\boxed{\text{are not independent}}$ because a nontrivial linear combination makes zero:

$$-\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = 0.$$

5. *All parts of this problem are about the matrix A. Suppose that its reduced row echelon form is*

$$R = \begin{bmatrix} 1 & 0 & * & * & 0 & 0 \\ 0 & 1 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where "$*$" is any number. [Careful! Answer about A, not R.] Does $A\mathbf{x} = 0$ have nontrivial solutions? Explain why or why not. Is $A\mathbf{x} = \mathbf{b}$ consistent for every vector $\mathbf{b}$? Expalin why or why not. Is column 3 of the matrix A the linear combination of its other columns? Explain.

The matrix equation $A\mathbf{x} = 0$ has the same solution as its reduced row echelon form $R\mathbf{x} = 0$. The system has free variables x_3 and x_4. They can be set to any values, say $x_3 = x_4 = 1$ and the pivot variables x_1, x_2, x_5 and x_6 are uniquely determined. Thus the system has a nonzero solution $x_1, \ldots, x_6$.

Moreover, for any vector $\mathbf{b}$, the RREF form of the augmented matrix $[A|\mathbf{b}]$ for the equation $A\mathbf{x} = \mathbf{b}$ can be row reduced to $[R|\tilde{\mathbf{b}}]$ where $\tilde{\mathbf{b}}$ corresponds to the $\mathbf{b}$ column after row operations. Since there is a pivot in every row of R, the system may be solved for any $\mathbf{b}$. Hence it is consistent for every $\mathbf{b}$.

Finally, since x_3 is a free variable we may set $x_3 = 1$, $x_4 = 0$ and determine the other variables of satisfying $R\mathbf{x} = 0$. Since R is row equivalent to A, the same $\mathbf{x}$ solves $A\mathbf{x} = 0$. In particular this gives a linear combination of the columns of A, namely

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + 1\mathbf{a}_3 + 0\mathbf{a}_4 + x_5\mathbf{a}_5 + x_6\mathbf{a}_6 = 0.$$

Thus we may solve for $\mathbf{a}_3$

$$\mathbf{a}_3 = -x_1\mathbf{a}_1 - x_2\mathbf{a}_2 - x_5\mathbf{a}_5 - x_6\mathbf{a}_6.$$

Hence $\mathbf{a}_3$ is a linear combination of the other columns of A.