Math 2270 § 4. **Second Midterm** Name: Solutions
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1. (a) *Detemine whether each of the given matrices is invertible. Use as few calculations as possible. Justify your answer.*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \text{Not Invertible.}$$

The first and last rows are equal. Row reduction will yield a zero row and so A is not row equivalent to I, thus not invertible.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix}, \qquad \text{Invertible.}$$

Row operations yield the matrix

$$B \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

which has a pivot in each row so B is row equivalent to I. Thus B is invertible.

$$C = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 0 & 3 & 1 & 0 \end{bmatrix}, \qquad \text{Not Invertible.}$$

The last column is twice the first so the columns of C are not linearly independent. Thus C is not invertible.

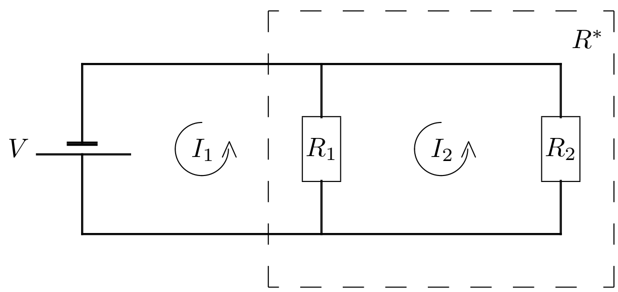
(b) *Let A be an $n \times n$ matrix. Without quoting Theorems 7 or 8, argue carefully why if there is a matrix D such that $AD = I$ then A is row equivalent to the identity matrix.*

We assume that there is a matrix D such that $AD = I$. Hence the system $A\mathbf{x} = \mathbf{b}$ may be solved for every $\mathbf{b} \in \mathbf{R}^n$. Indeed, by setting $\mathbf{x} = D\mathbf{b}$ we see that

$$A\mathbf{x} = A(D\mathbf{b}) = (AD)\mathbf{b} = I\mathbf{b} = \mathbf{b}.$$

The fact that $A\mathbf{x} = \mathbf{b}$ may be solved for every $\mathbf{b} \in \mathbf{R}^n$ says that there is a pivot in every row of the REF of A. Since A is an $n \times n$ matrix says that the RREF of A will be I. Thus A must be row equivalent to I.

2. *Write a matrix equation that determines the loop currents. Using an inverse matrix, solve for the currents in terms of the resistances* R_1, R_2 *and the voltage* V. *Compute the effective resistance* $R^* = V/I_1$ *of the dashed box (resistors in parallel.)*



Using Kirchhoff's Loop Law, we find that the loop currents satisfy

$$R_1(I_1 - I_2) = V$$
$$R_1(I_2 - I_1) + R_2 I_2 = 0.$$

Rewrite this system as the desired matrix equation,

$$\begin{pmatrix} R_1 & -R_1 \\ -R_1 & R_1 + R_2 \end{pmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}.$$

The solution may be given using the inverse matrix

$$\begin{aligned} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \begin{pmatrix} R_1 & -R_1 \\ -R_1 & R_1 + R_2 \end{pmatrix}^{-1} \begin{bmatrix} V \\ 0 \end{bmatrix} \\ &= \frac{1}{R_1(R_1+R_2) - R_1^2} \begin{pmatrix} R_1 + R_2 & R_1 \\ R_1 & R_1 \end{pmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix} \\ &= \frac{V}{R_1 R_2} \begin{pmatrix} R_1 + R_2 \\ R_1 \end{pmatrix}. \end{aligned}$$

The effective resistance is thus

$$R^* = \frac{V}{I_1} = \frac{R_1 R_2}{R_1 + R_2}.$$

Of course, this is the formula for resistors in parallel from freshman physics.

3. *Let* $A = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}$. *Find the inverse matrix* A^{-1}.

Augment the matrix by the identity and do row reductions.

$$\begin{bmatrix} 2 & 5 & 4 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 5 & 4 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & -2 & 0 \\ 0 & -2 & 5 & 0 & -1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 7 & -2 & 5 & 0 \\ 0 & 1 & -2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & -5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -16 & 40 & -7 \\ 0 & 1 & 0 & 5 & -12 & 2 \\ 0 & 0 & 1 & 2 & -5 & 1 \end{bmatrix}$$

So

$$A^{-1} = \begin{bmatrix} -16 & 40 & -7 \\ 5 & -12 & 2 \\ 2 & -5 & 1 \end{bmatrix}. \qquad \text{Check} \quad AA^{-1} = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} -16 & 40 & -7 \\ 5 & -12 & 2 \\ 2 & -5 & 1 \end{bmatrix} = I.$$

4. (a) *Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation that first rotates vectors by $+\pi$ radians and then mirror-reflects them across the $x_1 = x_2$ line. Find the standard matrix of T.* The standard matrix is determined by what the transformation does to the basic vectors, namely, $T(\mathbf{x}) = M\mathbf{x}$ where

$$M = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix}, \qquad \text{where} \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let R denote the rotation and S the rreflection, so $T = S \circ R$. Applying to $\mathbf{e}_1$ we see that

$$R(\mathbf{e}_1) = R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad S(R(\mathbf{e}_1)) = S\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$R(\mathbf{e}_2) = R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad S(R(\mathbf{e}_2)) = S\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence

$$M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(b) *Find an LU factorization of the matrix A. Check your answer, where $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \\ -5 & 0 & 2 \end{bmatrix}$*.Row reduce to the REF form of A, which will be our U. We may only use replacement operations where a lower row is replaced by the lower row plus a multiple of an upper row. We do not use exchange or scalar multiplication operations.

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \\ -5 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & 1 & 1 \\ 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{11}{3} \end{pmatrix} = A_2$$

$$\to \begin{pmatrix} 3 & 1 & 1 \\ 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} = U$$

where we have used $\frac{11}{3} - \frac{5}{2} \cdot \frac{5}{3} = -\frac{1}{2}$. We reconstruct the first column of L by dividing the first column of A by a_{11}

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{5}{3} & * & 1 \end{pmatrix}.$$

We reconstruct the second column of L by dividing the second column of A_2 by $\tilde{a}_{22}$, the middle entry of A_2,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{5}{3} & \frac{5}{2} & 1 \end{pmatrix}.$$

Check

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{5}{3} & \frac{5}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \\ -5 & 0 & 2 \end{pmatrix} = A.$$

where we have used $-\frac{5}{3} \cdot 1 + \frac{5}{2} \cdot \frac{5}{3} - 1 \cdot \frac{1}{2} = 2$.

5. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find matrices for which the statement fails.*

   (a) STATEMENT. *If the product of two* 2×2 *matrices* $AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *then* $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *or* $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

   FALSE. Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right)$. Then $AB = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

   (b) STATEMENT. *If for the* 2×2 *matrix* A *the equation* $A\mathbf{x} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ *can be solved then* A *is invertible.*

   FALSE. $A\mathbf{x} = \mathbf{b}$ has to be soluble for every $\mathbf{b}$ for A to be invertible. If we choose $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ then $A\mathbf{x} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ is has the solution $\mathbf{x} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ since $A\mathbf{x} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ but A is not invertible since its columns are linearly dependent.

   (c) STATEMENT. *If a* 2×2 *matrix satisfies* $A^2 = I$ *then a basis for* $\mathrm{Col}(A)$ *is* $\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$.

   TRUE. The equation $A^2 = I$ says that there is a matrix D such that $AD = I$, namely $D = A$ itself, so A is its own inverse and A is invertible. Hence the span of the columns of A, namely $\mathrm{Col}(A)$, is $\mathbf{R}^2$. Now the unit basic vectors $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ span $\mathbf{R}^2$. As they are also linearly independent, $\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$ is a basis of $\mathbf{R}^2 = \mathrm{Col}(A)$.

   (Not part of answer.) There are in fact many matrices besides $\pm I$ which are their own inverses. For example if x and y are any real numbers such that $y \neq 0$ then $A = \left(\begin{smallmatrix} x & 1/y \\ (1-x^2)y & -x \end{smallmatrix}\right)$ are all such, *i.e.*,

$$\begin{aligned} A^2 &= \begin{pmatrix} x & \frac{1}{y} \\ (1-x^2)y & -x \end{pmatrix} \begin{pmatrix} x & \frac{1}{y} \\ (1-x^2)y & -x \end{pmatrix} \\ &= \begin{pmatrix} x^2 + (1-x^2) & \frac{x}{y} - \frac{x}{y} \\ (1-x^2)xy - (1-x^2)xy & (1-x^2) + x^2 \end{pmatrix} = I. \end{aligned}$$