Math 2270 § 4. **First Midterm** Name: Solutions

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1. *Describe the solution of this system in parametric vector form. You must show all steps to receive full credit.*

$$\begin{aligned} 3x_2 + 2x_3 + 2x_4 + x_5 &= 4 \\ x_1 + 6x_2 + 5x_3 + 4x_4 + 2x_5 &= 8 \\ 3x_1 + 3x_2 + 5x_3 + 2x_4 + 2x_5 &= 4 \end{aligned}$$

Write the augmented matrix and perform row operations.

$$\begin{pmatrix} 0 & 3 & 2 & 2 & 1 & 4 \\ 1 & 6 & 5 & 4 & 2 & 8 \\ 3 & 3 & 5 & 2 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 6 & 5 & 4 & 2 & 8 \\ 0 & 3 & 2 & 2 & 1 & 4 \\ 3 & 3 & 5 & 2 & 2 & 4 \end{pmatrix} \quad \text{Swap } R_1 \text{ and } R_2$$

$$\to \begin{pmatrix} 1 & 6 & 5 & 4 & 2 & 8 \\ 0 & 3 & 2 & 2 & 1 & 4 \\ 0 & -15 & -10 & -10 & -4 & -20 \end{pmatrix} \quad \text{Replace } R_3 \text{ by } R_3 - 3R_1$$

$$\to \begin{pmatrix} 1 & 6 & 5 & 4 & 2 & 8 \\ 0 & 3 & 2 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{Replace } R_3 \text{ by } R_3 + 5R_2$$

x_3 and x_4 are free variables and can be set to any real value. Solving we find

$$\begin{aligned} x_5 &= 0 \\ 3x_2 &= 4 - 2x_3 - 2x_4 - x_5 = 4 - 2x_3 - 2x_4, \quad \text{hence } x_2 = \tfrac{4}{3} - \tfrac{2}{3}x_3 - \tfrac{2}{3}x_4. \\ x_1 &= 8 - 6x_2 - 5x_3 - 4x_4 - 2x_5 = 8 - 6\left(\tfrac{4}{3} - \tfrac{2}{3}x_3 - \tfrac{2}{3}x_4\right) - 5x_3 - 4x_4 - 2x_5 \\ &= 8 - 8 + 4x_3 + 4x_4 - 5x_3 - 4x_4 = -x_3. \end{aligned}$$

Thus the set of solutions is

$$\mathcal{S} = \left\{ \begin{bmatrix} -x_3 \\ \frac{4}{3} - \frac{2}{3}x_3 - \frac{2}{3}x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} : \text{where } x_3 \text{ and } x_4 \text{ are any real numbers.} \right\}$$

or in parametric vector form

$$\mathcal{S} = \left\{ \begin{bmatrix} 0 \\ \frac{4}{3} \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} : \text{where } x_3 \text{ and } x_4 \text{ are any real numbers.} \right\}.$$

2. *Let*

$$A = \begin{bmatrix} 4 & 1 & 5 & 2 \\ 3 & 2 & -1 & 2 \\ 2 & 2 & 8 & 2 \end{bmatrix}.$$

Do the columns of A span $\mathbf{R}^3$? Explain. Are the columns of A linearly independndt? Explain.

Perform row operations

$$\begin{pmatrix} 4 & 1 & 5 & 2 \\ 3 & 2 & -1 & 2 \\ 2 & 2 & 8 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & 8 & 2 \\ 3 & 2 & -1 & 2 \\ 4 & 1 & 5 & 2 \end{pmatrix} \text{Swap } R1 \text{ and } R3$$

$$\to \begin{pmatrix} 1 & 1 & 4 & 1 \\ 3 & 2 & -1 & 2 \\ 4 & 1 & 5 & 2 \end{pmatrix} \text{Divide } R1 \text{ by } 2$$

$$\to \begin{pmatrix} 1 & 1 & 4 & 1 \\ 0 & -1 & -13 & -1 \\ 0 & -3 & -11 & -2 \end{pmatrix} \begin{matrix} \\ \text{Replace } R2 \text{ by } R2 - 3R_1 \\ \text{Replace } R3 \text{ by } R3 - 4R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 1 & 4 & 1 \\ 0 & -1 & -13 & -1 \\ 0 & 0 & 28 & 1 \end{pmatrix} \text{Replace } R3 \text{ by } R3 - 3R_2$$

There is a pivot in each row. Hence, the columns of A span $\mathbf{R}^3$. There are more than three columns, so the columns cannot be linearly independent. Indeed, A has the free variable x_4, thus there is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$ which gives a notrivial dependency relation between the columns of A.

3. (a) *Let A be an $n \times n$ matrix. Show that $A\mathbf{x} = \mathbf{b}$ is consistent for all and $\mathbf{b} \in \mathbf{R}^n$ if and only if $A\mathbf{x} = \mathbf{0}$ has a unique solution.*

   $A\mathbf{x} = \mathbf{b}$ is consistent for all and $\mathbf{b} \in \mathbf{R}^n$ if and only if the REF of A has a pivot in each row. But for square $n \times n$ matrices, this accurs if and only if there is a pivot in each column. But this happens if and only if $A\mathbf{x} = \mathbf{0}$ has a unique solution.

   (b) *Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation such that*

$$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

   Solve $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$. *Find* $T\left(\begin{bmatrix} 1 \\ 9 \end{bmatrix}\right)$.

   First we solve $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$. The augmented matrix row reduces

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 9 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 8 \end{pmatrix}$$

   so $x_2 = 4$ and $2x_1 = 1 - x_2 = -3$ so $x_1 = -\frac{3}{2}$. This expresses the last vector as a linear combination

$$\begin{bmatrix} 1 \\ 9 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = -\frac{3}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

   It follows by linearity of T that

$$T\left(\begin{bmatrix} 1 \\ 9 \end{bmatrix}\right) = T\left(-\frac{3}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = -\frac{3}{2}T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) + 4T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$
$$= -\frac{3}{2}\begin{bmatrix} 0 \\ 4 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}.$$

4. (a) *Let* $T : \mathbf{R}^n \to \mathbf{R}^m$ *be a transformation. Define: the transforation* T *is* linear.

The transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is linear if for every $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ and every $c \in \mathbf{R}$ we have $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{u})$.

Let $N\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ 0 \end{bmatrix}$.

(b) *Give an example of a transformation* $N : \mathbf{R}^3 \to \mathbf{R}^2$ *such that* $N(\mathbf{0}) = \mathbf{0}$ *but* N *is not linear. Why is your transformation nonlinear?*

Let $N\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ 0 \end{bmatrix}$. We have $N(\mathbf{0}) = \mathbf{0}$. But neither condition of linearity holds. For example, for $\mathbf{u} = \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ Then

$$N(\mathbf{u} + \mathbf{v}) = N\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) = N\left(\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 16 \\ 0 \end{bmatrix}$$

$$\neq \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} = N\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) + N\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) = N(\mathbf{u}) + N(\mathbf{v}).$$

(c) *Let* $T(\mathbf{x}) = A\mathbf{x}$. *Is* $\mathbf{b}$ *in rhe range of* T *where*

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix}?$$

Forming the augmented matrix $(A|)$ and row reducing, we find

$$\begin{pmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \\ 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 0 & -5 & -15 \\ 0 & 1 & 3 \end{pmatrix} \quad \text{Replace } R_2 \text{ by } R_2 - 4R_1$$

$$\to \begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix} \quad \text{Replace } R_2 \text{ by } R2 + 5R3$$

The equations $A\mathbf{x} = \mathbf{b}$ are consistent. Thus $\mathbf{b}$ is in the range of T.

5. *Suppose that the row echelon form of the matrix A is R. Can $A\mathbf{x} = \mathbf{b}$ be solved for every $\mathbf{b} \in \mathbf{R}^4$? Explain why or why not. Suppose that the row echelon form of the matrix B is S. Is the solution of $B\mathbf{x} = \mathbf{0}$ unique? Explain why or why not. For the same B, let $T(\mathbf{x}) = B\mathbf{x}$. Is the range of T equal to $\mathbf{R}^6$? Explain why or why not. For "$*$" representing any number, the matrices are given by*

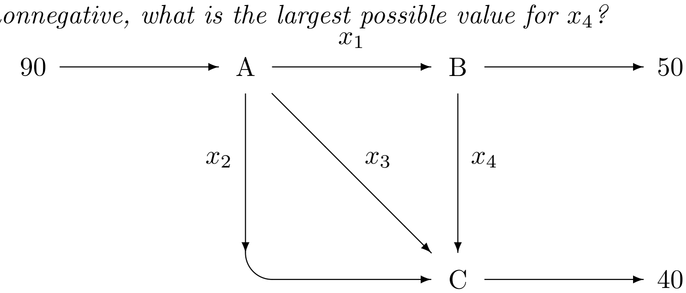
$$R = \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 2 & * & * & * \\ 0 & 0 & 0 & 0 & 3 & * \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & * & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 3 & * \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ can be solved for every $\mathbf{b} \in \mathbf{R}^4$? because its REF R has a pivot in every row.

The solution of $B\mathbf{x} = \mathbf{0}$, namely $\mathbf{x} = \mathbf{0}$, is unique since there are no free variables.

The range of $T(\mathbf{x}) = B\mathbf{x}$ does not equal $\mathbf{R}^6$. For some choices of $\mathbf{c} \in \mathbf{R}^6$, the row reductions $(B|c) \to (S|\tilde{\mathbf{c}})$ may not be consistent. The reduction may result in a $\tilde{\mathbf{c}}$ that is not zero in the fifth row, so there is no $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{c}$ for this $\mathbf{c}$.

6. *Find the general flow pattern of the network shown in the figure. Assuming that all the flows are nonnegative, what is the largest possible value for x_4?*



Let $x_1, \ldots, x_4$ be the flow rates (say in cars per minute) in the indicated directions of the segments. At each junction, there is a balance of cars entering the intersection and cars leaving the intersection each minute. Writing the balance equations,

$$\begin{aligned} (A) \qquad & 90 = x_1 + x_2 + x_3 \\ (B) \qquad & x_1 = x_4 + 50 \\ (C) \qquad & x_2 + x_3 + x_4 = 40 \end{aligned}$$

Moving variables to the left and constants to the right, we write the augmented matrix of the resulting system. Then we perform row operations.

$$\begin{pmatrix} -1 & -1 & -1 & 0 & -90 \\ 1 & 0 & 0 & -1 & 50 \\ 0 & 1 & 1 & 1 & 40 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 & 90 \\ 0 & -1 & -1 & -1 & -40 \\ 0 & 1 & 1 & 1 & 40 \end{pmatrix} \quad \begin{array}{l} \text{Replace } R_1 \text{ by } -R_1 \\ \text{Replace } R_2 \text{ by } R_2 + R_1 \end{array}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 0 & 90 \\ 0 & 1 & 1 & 1 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \text{Replace } R_2 \text{ by } -R_2 \\ \text{Replace } R_3 \text{ by } R_3 + R_3 \end{array}$$

Thus x_3 and x_4 are free and can take any values. Solving we find

$$\begin{aligned} & x_2 = 40 - x_3 - x_4 \\ & x_1 = 90 - x_2 - x_3 = 90 - (40 - x_3 - x_4) - x_3 = 50 + x_4. \end{aligned}$$

If all flow is in the direction of arrows, then $x_i \geq 0$. But this implies

$$x_2 = 40 - x_3 - x_4 \geq 0$$

which implies

$$x_4 \leq 40 - x_3 \leq 40$$

because $x_3 \geq 0$. This is the maximum possible x_4. In fact, taking $x_3 = 0$ and $x_4 = 40$ yields $x_1 = 90$ and $x_2 = 0$ we see that a valid flow takes the maximal x_4 value of 40.