Math 2270 § 2. **Fourth Midterm** Name: Solutions
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1. *Find all eigenvalues and eigenvectors. Show your work to get credit.*

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & -1 & 0 \\ 3 & 6 & 8 \end{bmatrix}$$

By expanding the second row the characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 2 & 3 \\ 0 & -1-\lambda & 0 \\ 3 & 6 & 8-\lambda \end{vmatrix} = (-1-\lambda) \begin{vmatrix} -\lambda & 3 \\ 3 & 8-\lambda \end{vmatrix} \\ &= -(\lambda+1)\left[(-\lambda)(8-\lambda) - 3\cdot 3\right] = -(\lambda+1)\left[\lambda^2 - 8\lambda - 9\right] \\ &= -(\lambda+1)(\lambda-9)(\lambda+1) = -(\lambda+1)^2(\lambda-9). \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = -1$ with algebraic multiplicity two and $\lambda_2 = 9$ with algebraic multiplicity one. We may find the eigenvectors by inspection

$$\begin{aligned} [\mathbf{0} \quad \mathbf{0}] = (A - \lambda_1 I)[\mathbf{v}_1 \quad \mathbf{v}_2] &= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{0} = (A - \lambda_2 I)\mathbf{v}_3 &= \begin{bmatrix} -9 & 2 & 3 \\ 0 & -10 & 0 \\ 3 & 6 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{aligned}$$

All eigenvectors are the nonzero vectors in eigenspaces

$$\mathcal{E}_1 = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{E}_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

2. *Find a matrix that diagonalizes A. Show that your matrix does the job.*

$$A = \begin{bmatrix} 7 & -3 \\ -1 & 5 \end{bmatrix}$$

First find eigenvalues and eigenvectors. The characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & -3 \\ -1 & 5-\lambda \end{vmatrix} = (7-\lambda)(5-\lambda) - (-3)(-1) \\ &= \lambda^2 - 12\lambda + 32 = (\lambda-4)(\lambda-8). \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 8$. We may find the eigenvectors by inspection

$$\begin{aligned} \mathbf{0} = (A - \lambda_1 I)\mathbf{v}_1 &= \begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \mathbf{0} = (A - \lambda_2)\mathbf{v}_2 &= \begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix}. \end{aligned}$$

The diagonalizing matrix is made of eigenvectors and the diagonal matrix has eigenvalues on the diagonal.

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}.$$

The easiest way to check is to compute

$$AP = \begin{bmatrix} 7 & -3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -24 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} = PD.$$

3. *Let $H = \operatorname{span}(S)$ be a subspace of $\mathbf{R}^4$. Show that $\mathbf{b}_1$ is in H. Find additional vectors $\mathbf{b}_2, \mathbf{b}_3, \ldots$ (as many as needed) so that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots\}$ is a basis for H. Explain.*

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}, \qquad \mathbf{b}_1 = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 6 \end{bmatrix}$$

Form the augmented matrix whose columns are S vectors along with $\mathbf{b}_1$ and reduce.

$$[A \,|\, \mathbf{b}_1] = \begin{bmatrix} 1 & 1 & 2 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 1 & 1 & 2 & 2 & 4 \\ 1 & 2 & 1 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The resulting system is consistent, so $\mathbf{b}_1 \in H$. Note that there are three pivots so $\dim H = 3$ and a basis for H is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$. Also the last column is a linear combination of the first two, $\mathbf{b}_1 = 2\mathbf{v}_1 + 2\mathbf{v}_2$ so we can take $\mathbf{b}_1$, one of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ as a basis, say

$$\text{A basis for } H \text{ is } \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 4 \\ 6 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

All three vectors are in H. The $\mathbf{b}_1$ and $\mathbf{b}_2$ are not multiples of each other, and $\mathbf{b}_3$ is not a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$ because the first and third components of $\mathbf{b}_3$ are unequal. Hence the set is a linearly independent set of three vectors in a three dimensional subspace H, thus is also spanning, hence a basis.

4. (a) *Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be two bases of a vector space V. Suppose that $\mathbf{b}_1 = -2\mathbf{c}_1 + 3\mathbf{c}_2$ and $\mathbf{b}_2 = \mathbf{c}_1 - 4\mathbf{c}_2$. Find the change of coordinates matrix matrix from $\mathcal{B}$ to $\mathcal{C}$. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = 6\mathbf{b}_1 + 5\mathbf{b}_2$.*

   We are given coordinates of the basic vectors and $\mathbf{x}$

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \qquad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

   The change of coordinates matrix is formed by columns that are coordinates

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}$$

   Applying the change of coordinates matrix we find

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \end{bmatrix}.$$

   (b) *Find a basis for the row space of A.*

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 2 & 1 \\ 3 & 6 & 4 & 2 \end{bmatrix}$$

Row reduce

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 2 & 1 \\ 3 & 6 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for Row(A) is

$$\left\{ \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -2 & -1 \end{bmatrix} \right\}.$$

5. *Consider two bases of* $\mathbf{R}^3$*.*

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \qquad C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

(a) *Find* $[\mathbf{w}]_{\mathcal{B}}$ *and* $[\mathbf{w}]_{\mathcal{C}}$*.*

(b) *Find the change of basis matrix* $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$*.*

(c) *Check that your* $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ *changes* $[\mathbf{w}]_{\mathcal{B}}$ *to* $[\mathbf{w}]_{\mathcal{C}}$*.*

Let

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad P_{\mathcal{C}} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

We have $P_{\mathcal{B}}[\mathbf{w}]_{\mathcal{B}} = \mathbf{w}$ so

$$[\mathbf{w}]_{\mathcal{B}} = (P_{\mathcal{B}})^{-1}\,\mathbf{w} = \frac{1}{1\cdot 0 - 1\cdot 1}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix},$$
$$[\mathbf{w}]_{\mathcal{C}} = (P_{\mathcal{C}})^{-1}\,\mathbf{w} = \frac{1}{1\cdot 1 - 0\cdot 2}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = (P_{\mathcal{C}})^{-1}P_{\mathcal{B}}$ we reduce the augmented matrix

$$[P_{\mathcal{C}}|P_{\mathcal{B}}] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} I & | & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{bmatrix}.$$

We find that the $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ does change $[\mathbf{w}]_{\mathcal{B}}$ to $[\mathbf{w}]_{\mathcal{C}}$, namely,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = [\mathbf{w}]_{\mathcal{C}}.$$

6. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find matrices for which the statement fails.*

(a) STATEMENT. $\left(\begin{smallmatrix} 4 & 2 \\ 1 & 3 \end{smallmatrix}\right)$ *is similar to* $\left(\begin{smallmatrix} 5 & 0 \\ 0 & 2 \end{smallmatrix}\right)$*.*
TRUE. The characteristic polynomial roots are the eigenvalues 5 and 2.

$$p_A(\lambda) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda-5)(\lambda-2)$$

Since the all eigenvalues are distinct, the matrix is diagonalizable, that is $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is similar to the matrix with eigenvalues as diagonals $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) STATEMENT. $\mathcal{F} = \{f : \mathbf{R} \to \mathbf{R}\}$, *the vector space of functions of the real numbers is finite dimensional.*

FALSE. The vector space of functions $\mathcal{F}$ contains the space of all polynomials $\mathbb{P}$. Any finite set $S \subset \mathcal{F}$ with n elements cannot span $\mathcal{F}$ because if it could, $\mathbb{P} \subset \text{span}(S)$. But the subspace $H = \text{span}\{1, t, t^2, \ldots, t^n\} \subset \mathbb{P}$ of dimension $n+1$ cannot satisfy $H \subset \mathbb{P} \subset \text{span}(S)$ because it does not satisfy the dimension inequality $\dim H \le \dim \text{span}(S) \le n$ which would have to be true for finite dimensional subspaces since $H \subset \text{span}(S)$.

(c) STATEMENT. *Suppose that* $p_A(\lambda) = 8\lambda^2 + 6\lambda^3 + \lambda^4$ *is the characteristic polynomial of* A. *Then* A *is invertible.*

FALSE. $p_A(0) = 0$ says zero is an eigenvalue so $A = A - 0I$ is singular. $p_A(\lambda) = \lambda^2(\lambda + 4)(\lambda + 2)$ is the characteristic polynomial of the singular matrix

$$A = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$